EFFECTIVE BUT UNDERUSED STRATEGIES FOR PROOF COMPREHENSION

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We present five strategies that mathematics majors can use to improve their proof comprehension. We argue these strategies are effective by presenting qualitative excerpts illustrating the ways in which the employment of these strategies helped four undergraduate students understand the proofs they were reading. Furthermore, we present results of a survey in which the majority of 83 mathematicians indicated they would like students to use these proof reading strategies. Finally, we argue that these strategies are underused by presenting results from another part of the survey, in which the majority of 175 mathematics majors claimed not to employ these strategies.

Keywords: Reasoning and Proof, Advanced Mathematical Thinking, Post-Secondary Education

Introduction

In lamenting the quality of instruction in university mathematics courses, Davis and Hersh (1981) contended that “a typical lecture in advanced mathematics… consists entirely of definition, theorem, proof, definition, theorem, proof, in solemn and unrelieved concatenation” (p. 151). While such a description probably exaggerates the frequency with which proofs are presented in mathematics lectures, it is the case that much of the teaching in advanced mathematics courses consists of students observing the proofs that their professors present to them (e.g., Weber, 2004; Fukawa-Connelly, 2012; Mills, 2011).

To date, research in undergraduate mathematics education on proof reading generally falls into one of two categories. Some researchers have sought to examine the types of arguments that convince mathematics majors of the truth or falsity of mathematical statements. These researchers typically present students with arguments based on different types of evidence (e.g., empirical evidence, perceptual evidence) and ask students to evaluate the persuasiveness of these arguments (e.g., do you find this argument personally convincing?). The researchers use students’ evaluations as a lens to determine their proof schemes (e.g., Harel & Sowder, 1998; Segal, 2000). In a second category of research, researchers present mathematics majors with purported proofs and ask them to determine if the presented arguments are mathematically valid (e.g., Inglis & Alcock, 2012; Selden & Selden, 2003). This research reveals students’ difficulty with validation tasks as well as deficiencies in the processes they use to validate proofs.

In this paper, we introduce a third way to investigate mathematics majors’ proof reading by asking: what strategies can undergraduate students use to better comprehend proofs? The common pedagogical practice of presenting proofs to mathematics majors is based on the assumption that students can learn mathematics from reading them. Both mathematicians and mathematics educators question whether this assumption is true (e.g., Conradie & Firth, 2000; Cowen, 1991; Rowland, 2001). If mathematics majors do not gain understanding from the proofs that they read, it is natural to ask what strategies they can use to improve their comprehension. The purpose of this paper is to address this issue.

Theoretical Perspective

Our theoretical perspective builds upon answers to two related questions found in the literature: why do we present proofs to mathematics majors, and what does it mean for students to understand a proof? Central to the question of how mathematics majors should read proofs is why they are asked to read these proofs in the first place. Mathematicians read proofs published in journals for a variety of reasons, including understanding why theorems are true (de Villiers, 1990; Hanna, 1990) and finding techniques that would be useful for them to prove other theorems in their own research (e.g., Rav, 1999; Weber & Mejia-Ramos, 2011). Many mathematics educators argue that proof should play a similar role in mathematics classrooms. Proofs that merely convince students that a theorem is true are thought to have little pedagogical value (Hanna, 1990; Hersh, 1993), especially as many students are convinced a theorem is true if their teacher or textbook tells them that this is so (Harel & Sowder, 1998). Pedagogical proofs should also provide students with insights such as explanations of *why* the proven result holds true (Hanna, 1990; Hersh, 1993) and other related goals. Although secondary teachers do not appreciate these broader functions of proof (Knuth, 2002), interviews with mathematicians reveal that they present proofs to students for these purposes (Yopp, 2011; Weber, 2012).

Mejia-Ramos et al (2012) put forth a model delineating what it would mean for a student to understand a proof in advanced mathematics. In this model, a proof can be understood either locally as a series of individual deductions, or holistically based upon the ideas or methods that motivate the proof in its entirety. At a local level, understanding a proof would be comprised of (a) knowing the meaning of the terms and statements within the proof, (b) being able to justify how new assertions in a proof followed from previous ones, and (c) understanding how the assumptions and conclusions related to the proof method being used. In this model, the holistic understanding of a proof consists of being able to: (a) provide a summary of the proof that emphasizes its high level goals, (b) apply the methods of the proof in other situations to prove new theorems, (c) break the proof into sub-proofs, and (d) apply the methods of the general proof to a specific example.

As a final note, the ways that proofs are conventionally presented in advanced mathematics mean several dimensions of the Mejia-Ramos et al's proof comprehension model are not transparent. For instance, proofs often contain gaps where the reader has to infer what principles are being used to deduce new statements from previous ones (Weber & Alcock, 2005). The linear deductive manner in which proofs are written often inhibits students from seeing the higher-level ideas of the proof, or how the proof could be broken into its main parts (Leron, 1983). Hence, understanding a proof involves much more than making a literal translation of each statement within the proof, and requires active construction on the part of the reader. In this paper, we describe strategies that mathematics majors can use to facilitate such construction.

Methods

Rationale of this Study

This study employs both qualitative and quantitative methods. First, in a qualitative study, four successful mathematics majors were observed thinking aloud while reading six mathematical proofs. These videotapes were analyzed to identify proof reading strategies that were used by these students to facilitate comprehension. Excerpts from these videotapes illustrate the ways in which these strategies were effective. After these strategies were identified, a quantitative survey study was designed to determine the extent to which: (a) mathematics majors claim to use these strategies, and (b) university mathematics professors desire that their students use these strategies. This survey data provided evidence that these strategies were mathematically desirable (from the viewpoint of mathematicians), but were not used by many mathematics majors (by their own self-report).

Qualitative Study of Mathematics Students Reading Proofs

Participants. Four seniors who were mathematics majors and prospective teachers agreed to participate in this study. These participants were chosen due to their success in their content-based mathematics courses (suggesting they used productive proof reading strategies), and because we found them to be thoughtful and articulate.

Materials. Participants were asked to read six proofs. For the sake of brevity, they are not included in this report but can be found in Weber & Samkoff (2011). The proofs were chosen such that the mathematical content relied on calculus and basic number theory (so lack of content knowledge would not inhibit proof comprehension) and the proofs employed an interesting technique with which the participants would not have extensive experience. For each proof, we generated comprehension questions using Mejia-Ramos et al's (2012) proof comprehension model described above.

Procedure. Participants met in pairs with the first author for a task-based interview. The first pair of students was assigned the pseudonyms Kevin and Tim, and the second pair was assigned Caleb and Derek. Participants were asked to "think aloud" as they were videotaped reading the proofs. They were given an individual proof and asked to read the proof until they understood it. Once this understanding was attained, the researcher took the proof and asked them to complete proof assessment questions. The interviewer then repeated this process with a new proof. This continued until all six proofs were read. Both pairs of students answered nearly every question correctly (which was uncommon in our proof reading studies), indicating that the pairs of students read the proofs effectively.

Analysis. Since we were looking to identify new strategies, and given that the proof comprehension literature is limited, we used an open-coding scheme in the style of Strauss and Corbin (1990) to identify new strategies. In the first pass through the data, we identified the strategies that were of interest to us. In the second pass, we coded explicitly for each use of those strategies.

Quantitative Survey Study

Survey items. For each strategy that we identified in the qualitative study, we created two survey items. For the mathematics majors, we generated a two choice survey item. Choice A claimed that they frequently used the identified strategy when they read proofs, while Choice B indicated that they did not regularly use the strategy. The survey item asked participants to choose whether they agreed with Choice A, Choice B, or were neutral between the two using a five-point Likert scale. The survey item for the mathematicians was similar, except they were asked if they would prefer that their students used the strategies, not that they would use the strategies themselves. For example, one of the identified strategies consists of trying to prove the theorem before reading its proof. For the students, the two choices in the survey item were:

A. When reading a theorem, I usually try to think about how I would prove the theorem before reading its proof.
B. I do not usually try to prove a theorem before reading its proof. A reason for reading the proof is to see why the theorem is true.

For the mathematicians, the two choices were:

A. When reading a proof of a theorem, I would prefer if mathematics majors think about how they might prove the theorem themselves before reading the proof.
B. I would prefer that mathematics majors not try to prove a theorem themselves before

reading its proof. A reason for reading the proof is to see why the theorem is true.

Participants. The participants in this study were invited to take part in an Internet survey. The participants from this study were conducted from 50 large state universities in the United States. The mathematics department secretaries at these universities were sent an e-mail asking them to forward e-mails to their faculty and their students. Mathematicians were invited to participate in a survey about proof reading with a link to a website for participation. Mathematics majors were sent separate request to participate in a proof reading study with a link to a separate website. Upon reaching the website, participants were asked for demographic information. For mathematicians, one question included whether they had ever taught a proof-oriented course. For students, one question included whether they had ever completed a proof-oriented course. If participants answered no to these questions, their responses were not included for analysis. 175 mathematics majors and 83 mathematicians met this requirement and completed the survey. Because the data were completely anonymous, we do not know which secretaries sent the requests to their faculty or students nor what percentage of faculty or students agreed to participate.

Results

We describe the five strategies that we observed.

Strategy #1: Trying to prove a theorem before reading its proof

For each of the six proofs that Kevin and Tim read, they would first try to prove the theorem before reading its proof. For instance, Kevin and Tim were handed a proof of the claim "$4x^3 - x^4 + 2\sin x = 30$ has no solutions". Immediately after reading the claim, Kevin and Tim began analyzing why the claim was true:

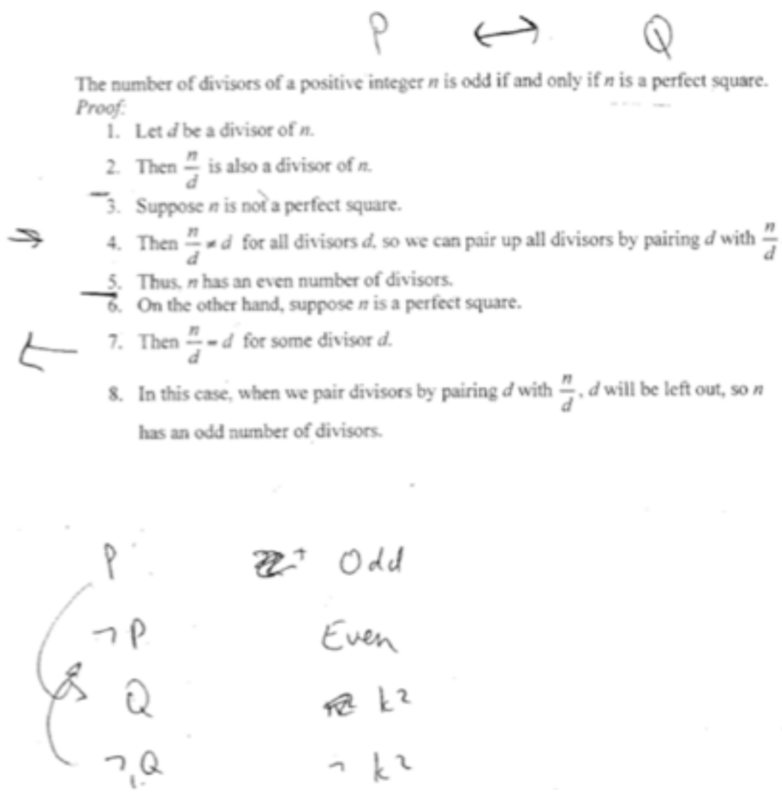
Tim: As x gets really big, it gets dominated by the negative x to the fourth term. And it's a parabola going down basically and it's going to get modulated a little bit.
Kevin: Right. And sine of x is…
Tim: Periodic.
Kevin: It is periodic so that wouldn't really affect it too much… out of the two functions, *f(x)* is the trumping one.
Tim: So in the long run, it's going…
Kevin: It's really *f(x)* that matters.
Tim: And the question is, does it reach 30.

It is important to note that Kevin and Tim were not only trying to understand the statement, they were also trying to understand why the statement was true. While they did not successfully produce a proof, their efforts at this stage appeared to help them comprehend the proof that they read. The proof itself had the same high-level ideas that Kevin and Tim highlighted: finding the bounds of $2\sin x$ and $4x^3 - x^4$. The details in the actual proof, such as using differentiation to find the critical points of $f(x)=4x^3 - x^4$, can be seen as supporting these high-level goals. In this sense, when reading the proof, Kevin and Tim did not view the proof as an aimless series of inferences and calculations, but rather saw the proof as satisfying two high level goals—finding the bounds of $2\sin x$ and $4x^3 - x^4$. Hence, trying to prove the theorem before reading the proof facilitated Tim and Kevin's ability to "structure" the proof (in the sense of Leron, 1983) and provide a summary of it. Other transcripts show how trying to write a proof helped Tim and Kevin understand the proof technique (e.g., proof by contraposition) being employed in the proofs that they read and identifying techniques that were novel to them in the proof that might be useful in other settings.

Using the survey item described in the previous section, we found that 88% of the survey mathematicians chose choice A, desiring students use this strategy, but only 31% of the mathematics majors chose choice A, suggesting most mathematics majors do not use the strategy.

Strategy #2: Comparing the assumptions and conclusions in the proof with the proof technique being used and Strategy #3: Breaking a longer proof into parts or sub-proofs

We illustrate both strategies by showing how Tim and Kevin read the proof presented in Figure 1:



P $\leftrightarrow$ Q

The number of divisors of a positive integer n is odd if and only if n is a perfect square.

Proof:

1. Let d be a divisor of n.
2. Then $\frac{n}{d}$ is also a divisor of n.
3. Suppose n is not a perfect square.
4. Then $\frac{n}{d} \neq d$ for all divisors d, so we can pair up all divisors by pairing d with $\frac{n}{d}$.
5. Thus, n has an even number of divisors.
6. On the other hand, suppose n is a perfect square.
7. Then $\frac{n}{d} = d$ for some divisor d.
8. In this case, when we pair divisors by pairing d with $\frac{n}{d}$, d will be left out, so n has an odd number of divisors.

Figure 1: Tim and Kevin's Employment of Strategies 2 and 3

Kevin: From [lines] two to four, it's doing the proof by contradiction. Suppose *n* is not a perfect square. […]

Tim: Suppose *n* is not a perfect square. So you're saying, suppose not this?

Kevin: For which one?

Tim: So you're saying, suppose *n* is not perfect, right, and that's the opposite of the right side.

Kevin: Cause it's dichotomous. If it's not a perfect square, then it's even. So therefore if P implies Q, then not Q implies not P, right? Contrapositive.

Tim: So we're saying this is P [writes P above "the number of divisors of a positive integer n is odd], this is Q [writes Q over "n is a perfect square"].

Kevin: OK sure. So P is a positive integer, right? Positive integer?... Right?... Positive integer?

Tim: Is odd?

Kevin: Right. Sorry. P is odd, right? So then not P would be even. So Q would be a perfect square, k squared. And not Q would not be k^2 or whatever. So here, [referring to lines 3-5 of

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Proof 5] it shows not Q implies not P.

Tim: And this part of the proof ends at 5 [Tim draws horizontal dashed between lines 2 and 3 and between lines 5 and 6 to partition the two parts of the proof]

Kevin: So it shows not Q implies not P and therefore P implies Q. So it's proving it forwards. [Tim rights a right arrow next to lines 3, 4, and 5]. It's the forwards way. [Reading line 6] On the other hand, suppose *n* is a perfect square. So now it's going to prove the backwards.

Selden and Selden (2003) and Weber (2010) found that students would often accept a proof of a conditional statement as valid in cases where the proof began by assuming the conclusion of the conditional statement and deduced the antecedent—i.e., the argument proved the converse of the statement. Both research teams took this as evidence that participants did not attend to the proof framework of the proofs that they were reading. The excerpt above illustrates that Kevin and Tim explicitly attend to the assumptions and conclusions of the proof to understand how a valid proof technique is being employed. The surveyed students were presented with these choices:

A. When I read a proof, I first consider is what is being assumed, what is being concluded, and what proof technique is being used.

B. When I read a proof, I first consider how each new statement can be derived from previous statements.

Only 33% of the mathematics majors preferred A, but 64% of the mathematicians preferred A for the analogous item (not presented here for the sake of brevity). In the transcript and figure above, we also see how Kevin and Tim broke the proof into two parts, lines 3 through 5, and lines 6 through 8. This was something that both pairs of students regularly did. In the survey, the mathematics majors were given the following two choices.

A. When I read a long proof, I try to break it into parts or sub-proofs.

B. When I read a long proof, I do not break it into parts but try to understand how each line follows from previous assertions.

Less than half (38%) of the mathematics majors chose A, but 88% of mathematicians chose A for the analogous items.

Strategy #4: Comparing the proof approach to the one's own approach

After reading the proof that "$4x^3 - x^4 + 2\sin x = 30$ has no solutions", Caleb and Derek noted that this is not how they would approach the problem. (The proof proceeded by showing $2\sin x \leq 2$ and $4x^3 - x^4 \leq 27$ for all real x).

Caleb: Definitely not the way I would have gone when I looked at it…

Derek: Not even close.

Caleb: Yeah, I would not have broken it down like that.

Derek: I would not have broken it down like that at all. How I would have solved this, I have no idea.

Caleb: Well when you start putting in trigonometric functions.

Derek: Yeah, that's when it starts getting a little hairy […] I probably would never have ever thought of this but from my background in calculus, it makes sense to me.

Caleb: Yeah, yeah. It's clever [laughs]

Derek: Yeah, it's clever. I never would have dreamed of coming up with something like this.

After reading all six proofs, Caleb and Derek were asked if comparing a proof strategy to one's own approach was something they regularly did. Both agreed that it was. When asked why they did this, Derek said: "it gives you new techniques to solve proofs like that".

The surveyed mathematics majors were given the following choices:

A. When I read a proof, I compare how the method used in the proof compares to the method I would use to prove the theorem.
B. When I read a proof, I try not to consider how I might approach the proof, but focus on what method was used in the actual proof.

Few mathematics majors selected Choice A (26%) but most mathematicians (87%) selected A when given the analogous choices.

Strategy #5: Using an example to understand a confusing inference

When both pairs of students were confused by a statement in number theory, they interpreted the general statement in terms of specific numbers in order to understand what the statement was asserting, and why it was true. For instance, one proof contained the line, "Let d be a divisor of n. Then n/d is also a divisor of n". Tim and Kevin were confused as to why the second claim should be true until Kevin said, "So, say 7 is a divisor of 21. So to get the other factor, it's 3". After this, the pair were satisfied that every product of a number contains two factors and if d was a factor, n/d would be the other factor. The surveyed students read the following choices:

A. When I read a new assertion in a proof, I sometimes check whether that assertion is true with a numerical example.
B. When I read a new assertion in a proof, I try to see how that new assertion is a logical consequence of previous statements. I do not check assertions with specific examples since you cannot prove by example.

A minority (43%) of mathematics majors chose A, but 77% of mathematicians chose A when given the analogous items, indicating most mathematicians would like students to use this strategy.

Discussion

This paper presents five strategies that mathematics majors can use to improve the comprehension of the proofs that they read. For each strategy, we illustrate how it can be useful with an episode of a pair of strong students reading proofs for comprehension. We then presented quantitative data indicating the majority of mathematicians would like their students to use these strategies, but less than half the students agree to regularly using these strategies. This suggests that if mathematics majors could be taught how to apply these strategies and were encouraged to use them, their proof comprehension performance might improve.

There are several caveats worth mentioning. There were only a small number of students (4) reading a small number of proofs (6) in the qualitative study. It is probable that interviewing other strong students or using other proofs would have evinced other strategies as well. Hence, we make no claims that the five strategies that we highlighted are exhaustive. Furthermore, the survey for students was self-report. It is possible that some mathematics majors might not have reported using these strategies, maybe because they were not aware that they did so. If so, explicitly naming and highlighting these strategies may help these students leverage them more effectively.

Finally, the link between identifying strategies and teaching students to use them is not straightforward. The use of these strategies likely relies on students' conceptual knowledge (that may or may not be adequate). Further, these strategies are descriptive and this description might not be detailed enough for students to use them effectively (see Schoenfeld, 1985). We are currently conducting teaching experiments, both to refine these strategies so they are useful to mathematics majors and to find environments and activities that foster the understanding, appreciation, and adoption of these strategies.

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