CONVICTION AND VALIDITY: MIDDLE SCHOOL MATHEMATICS TEACHERS' PROOF EVALUATIONS

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This paper examines 55 middle school mathematics teachers' proof evaluations and their perceived criteria for what constitutes convincing arguments and valid mathematical proofs. The results suggest that the majority of teachers find: (1) a number-based argument to be neither convincing nor a proof, and (2) an algebraic argument and a visual argument to be both convincing and a proof. The results also suggest that many teachers perceive generality as the major criterion in order for arguments to be considered as convincing and a proof.

Keywords: Reasoning and Proof, Middle School Education, Convincing Arguments

Introduction

Both the Common Core State Standards for Mathematics (Common Core State Standards Initiative [CCSSI], 2010) and the Principles and Standards for School Mathematics (National Council of Teachers of Mathematics [NCTM], 2000) suggest that students at all grade levels should be able to evaluate and to understand mathematical arguments. Successfully responding to this call requires that teachers have adequate understandings of mathematical proof. Research on mathematics teachers' competencies in understanding and evaluating proofs, however, has documented that teachers have difficulty recognizing given arguments as mathematical proofs (e.g., Knuth, 2002; Martin & Harel, 1989; Morris, 2002). We suggest that examining teachers' criteria for what constitutes convincing arguments and mathematical proofs may help teachers develop a better understanding of valid proof structures (e.g., proof by induction). Yet, to date, little is known about what arguments middle school mathematics teachers find convincing and accept as mathematical proofs. To address this research gap, this study investigates middle school mathematics teachers' reasons for deciding whether or not a given argument is convincing and whether it is a valid mathematical proof.

Theoretical Background

To understand middle school mathematics teachers' criteria for what constitutes a convincing argument and a valid mathematical proof, we conceptualize characteristics of convincing arguments and mathematical proofs using Knuth's (2002) taxonomies that he used to characterize experienced high school mathematics teachers' views on proof. Knuth (2002) found with in-service high school mathematics teachers the following criteria for evaluating whether an argument is convincing: concrete features (i.e., the use of numbers or diagrams), familiarity, sufficient level of detail, generality, shows why, and valid method (i.e., proof by induction). He also found that valid methods, mathematically sound (i.e., the argument showed the underlying mathematics), sufficient detail (i.e., the argument included all of the steps), and knowledge dependent (i.e., the teacher's understanding of the mathematics presented in the argument) are four major criteria that teachers used to consider an argument to be a valid proof. Given that personal criteria have the potential to influence a teacher's evaluation of whether an argument is convincing and/or a valid proof, we examine the types of arguments that teachers found convincing and acceptable along with the criteria they used to justify their evaluation.

Methods

A total of 55 teachers from nine public school districts in the Northeast region of the United States completed an online survey. The survey presented teachers with one conjecture and three arguments for the same conjecture—a number-based argument, an algebraic argument, and a visual argument (see Figure 1). Teachers were asked to: (1) indicate how convincing they found the given argument using a “very convincing,” “convincing,” “somewhat convincing,” or “not at all convincing” scale; (2) decide if each given argument was a mathematical proof using a “yes” or “no” response; and (3) describe what criteria they used to make their decisions. This paper focuses on teachers’ proof conviction and validation. A response was coded as convincing if a teacher indicated the argument was “very convincing” or “convincing” and was coded as not convincing if a teacher indicated the argument was “somewhat convincing” or “not at all convincing.” The teachers’ criteria for convincing arguments and mathematical proofs were analyzed based on the theoretical background described previously. As the data were examined, different types of criteria emerged in participants’ written responses. The characteristics for convincing arguments and proofs with a representative example are listed in Table 1.

Rob comes up with the following conjecture: The sum of any two even integers is an even integer.

Jodi’s answer

2+4=6
4+6=10
1002+856=1858
20,147,938+3,144,028=23,291,966
Thus, Jodi says Rob’s conjecture is true.

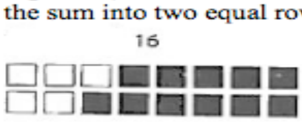
Matt’s answer

If a and b are even integers, then a and b can be written $a=2m$ and $b=2n$, where m and n are other integers.
So $a+b=2m+2n$. We can factor out a 2 and then get $a+b=2(m+n)$. Since $(m+n)$ is an integer, $a+b$ must be even.
Thus, Matt says Rob’s conjecture is true. (Adapted from Smith & Stein, 2011, p. 46)

Teresa’s answer

If I take the numbers 6 and 10 and organize the counters as shown, you can see the pattern.

You can see that each number can be made into two equal rows. When you put the sets together (add the numbers), the blocks are still two equal rows. So, since you can arrange the sum into two equal rows, it must be an even number.



Thus, Teresa says Rob’s conjecture is true. (Adapted from Smith & Stein, 2011, p. 46)

Figure 1: The Conjecture with Three Arguments

Results and Discussion

Which Arguments are Convincing

Thirty-nine of the 55 participants did not find Jodi’s answer (the number-based argument) to be convincing, while the other 16 teachers judged it as convincing. This result is not in alignment with prior research showing that the majority of teachers are convinced by arguments based on specific examples (e.g., Knuth, 2002). Thirty-five teachers identified Matt’s answer (the algebraic argument) and Teresa’s answer (the visual argument) to be convincing, which are consistent with Knuth’s (2002) findings that some in-service high school mathematics teachers were convinced by arguments with algebraic or visual features. The most prevalent criterion for convincing arguments was the use of generality (41 cases out of 176), as many teachers indicated a belief that a convincing argument should have generalization for all cases.

It is interesting to note that there were 24 cases in which the teachers were convinced by arguments based on Table 1: Characteristics of Convincing Arguments and Mathematical Proofs to be true. We also found that a few teachers thought that Matt should test specific numbers to support the argument he had made using symbols, which suggests that they did not seem to understand that a generalized argument provides certainty for all cases. However, the findings only represented 22 cases that the teachers used for convincing arguments. In addition, there were 18 cases in which the teachers considered if arguments were easy to understand to be convincing. This finding suggests that teachers should have certain knowledge of mathematics in order to understand arguments.

Table 1: Characteristics of Convincing Arguments and Mathematical Proofs

Category	Definition	Representative Example of Convicting Arguments	Representative Example of Proofs
Use of Generality	An argument has/has not shown the certainty for all cases.	Only 3 specific examples, out of an infinite number of possibilities. There was no generalization for all cases.	It (Matt's answer) proves the statement for the general case.
Use of Numerical Representations	An argument has/has not provided a variety of numbers.	Jodi shows several various examples that prove the statement.	I tried plugging in Matt's proof using random #s and couldn't get it to prove.
Use of Counterexamples	An argument has/has not included a counterexample to disprove the statement.	Jodi only used whole positive numbers. She needed to look for a counterexample.	She (Jodi) did not eliminate the chance of missing an exception - it only takes one "wrong" solution to invalidate the postulate.
Use of Visual Representations	An argument has/has not used visual diagram.	I gave Teresa very convincing because she used a diagram to explain her reasoning. I think using pictures to organize her thinking is a very effective and clear way to convince someone she is correct.	I read through her thinking and she used pictures as a proof.
Use of Clarity	An argument is/is not easy to follow.	It (Teresa's answer) is a clear and understandable argument.	I couldn't follow it (Matt's answer) so I thought no, but I realize that it is not a valid answer
Use of Mathematical Facts	An argument has/has not included known definitions or theorems.	(Matt) [u]sed mathematical evidence such as the fact that an even number is divisible by two and the distributive property	If it (Jodi's answer) showed logical reasoning backed up with known (proven) mathematical concepts, it did not.
Use of Logical Structures	An argument has/has not (1) stared with the givens or (2) included all steps that follow logically from previous assertions.	He (Matt) used logical steps that are beyond dispute to reach a conclusion.	She (Jodi) did not use a series of logical steps that are beyond dispute to reach a conclusion.
Use of Symbolic Manipulations	An argument has/has not (1) presented algebraic symbols or formulas or (2) manipulated symbols correctly.	(Matt's answer) [u]s[ed] algebraic variables instead of specific numbers proves the argument in more general terms.	She (Jodi) needs to state some kind of rule(s) and then base her statement off of that.
Use of Explanations	An argument has/has not provided a clear explanation.	Matt is adding variables, but not explaining why he can write that a=2b and so on. He is multiplying by two and does not explain why that makes numbers even.	Teresa is clearly showing two even numbers will always equal an even number. She proves this by showing us why.
Use of Similarity	An argument has/has not used the similar way as the teacher teachers/produces a proof.	Jodi's argument is similar to my own in that she used real math facts to prove she was correct.	We do a similar activity in our math class using tiles to create rectangles. Teresa demonstrated this using the same strategy we teach our students.
Other	A teacher's response is not clear.	She needed more proof!	I don't know.

Which Arguments are Mathematical Proofs

Thirty-nine of the 55 participants did not accept Jodi's answer using specific numbers as an acceptable proof, while the remaining teachers had a different belief. This finding is in contrast to previous results suggesting that pre-service elementary and secondary mathematics teachers tend to view number-based examples as valid proofs (e.g., Martin & Harel, 1989; Goetting, 1995). Thirty-seven teachers found Matt's answer using algebra to be a proof, and 27 teachers judged Teresa's answer using a visual representation to be a proof. The former result is consistent with Raman's (2002) findings that many undergraduate students believe that arguments including mathematical symbols are proofs. The latter finding is reminiscent of Weber's (2010) suggestion that diagrammatic arguments are acceptable forms of proofs because they provide insight into why statements are true. We also found that there were 31 cases in which the teachers judged the argument to be a mathematical proof because it had or had not used algebraic rules or mathematical symbols. These results are not entirely surprising considering the literature on undergraduate students' competencies in evaluating proofs suggesting that they are inclined to accept arguments with mathematical symbols as proofs (e.g., Raman, 2002). In order to explore middle school mathematics teachers' views on what constitutes proof, mathematics teacher education programs might consider providing more opportunities for teachers to evaluate various types of arguments.

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