TRACING ARIEL'S GROWTH IN ALGEBRAIC REASONING: A CASE STUDY

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We report the results from a case study of Ariel, a middle-school participant in a 3-yr longitudinal study of the development of understanding of mathematical ideas. We focus on Ariel's use of arithmetic knowledge in finding a rule for his solution to a problem task that takes the form of a composite function. Over a year later, after being introduced to the technical algebra language and formal notation, Ariel revisits the problem and offers a general, closed-form solution. When re-solving the problem his language is more precise as he connects meaning to the symbols. Viewing a video of his earlier solution, Ariel acknowledges the correctness of his earlier work and indicates that the earlier solution was not generalizable.

INTRODUCTION

In this study, we examine the development of algebraic reasoning of Ariel, a participant in a 3-year, after-school, informal mathematics program. We focus on Ariel's engagement with early algebra ideas, particularly his work on the Ladders Problem (Davis, 1964). The task requires that the student construct a rule to predict the number of Cuisenaire rods needed to build a ladder with varying number of rungs. We report Ariel's problem solving in sessions, fifteen months apart - grade 7 and the end of grade 8 solving the same problem. The research questions guiding this study are: (1) How does Ariel make use of his knowledge of arithmetic and algebra to build his solutions to the Ladders Problem? and (2) What change, if any, was there in Ariel's problem solving from grade 7 to 8?

THEORETICAL FRAMEWORK

While research has shown that understanding the concept of a function is essential for success in other areas of mathematics (Carlson, 1998; Rasmussen, 2000), students continue to struggle learning the concept (Vinner and Dreyfus, 1989). Yet, there is promising work that supports the claim that young children, who are engaged in problem-solving activities designed to elicit justifications for their solutions, can develop understanding of fundamental algebraic ideas such as function (Maher, Powell & Uptegrove 2010; Kieran, 1996; Yerushalmy, 2000; Kaput, Carraher, & Blanton, 2008). As early as 1985, Davis advocated the introduction of algebra to elementary age students, some even as young as grade 3. He argued that the idea of function can be built intuitively by learners as they engage in explorations of problems that called for the identification of increasingly more challenging patterns, and that students can build the conceptual idea before formal notation is introduced. In his work, Davis has offered sets of tasks for student exploration, capturing on video the problem solving of young children as they successfully find solutions that can be expressed with linear, quadratic and exponential functions (Mayansky 2007; Giordano, 2008). Also, examples of

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children providing verbal expressions of algebraic function before learning to write the rules in symbolic form have been reported by Bellisio and Maher (1998). This study seeks to extend earlier work by examining in detail how one student builds an understanding of the linear function concept and represents his understanding of the basic algebra ideas underlying its construction

METHODOLOGY

Setting

This study is a part of a larger three-year, research project, Informal Mathematics Learning (IML), a partnership program between the Robert B. Davis Institute for Learning (RBDIL) and an economically depressed urban school district in the USA. The case study was conducted with middle school aged students (11 to 13 years). A primary goal was to investigate how mathematical ideas and ways of reasoning by students were developed over time in an informal, after school environment. The content strand reported here is algebra. Ariel and other members of his cohort, were introduced to functions by engaging in Guess My Rule activities

The algebra strand began at the end of the second year of the program. Ariel participated in six group algebra sessions over 3 weeks and one interview about his experiences in the algebra domain 15 months later. Each session lasted 60- to 80 minutes. At all of the sessions, the student participants worked on open-ended problem solving tasks that challenged them to build the rule that described the function.

Task

For this report, we focus on a task that asks students to determine how many light green Cuisenaire rods (Figure 1) would be needed to build a ladder with different number of rungs. The shortest ladder has only one rung and can be built with 5 light green Cuisenaire rods. A two-rung ladder would be modelled using 8 light green rods. It was of interest to see if students could provide a general solution to the problem. The problem was presented as follows:

<u>The Ladders Problem:</u> Build a rod model to represent a 3-rung ladder. How many rods did you use? How many rods would you need to build a ladder with 10 rungs? How could you represent the number of rods needed if you were to build a ladder with any number of rungs? Justify your solution.

Figure 1: Picture of Cuisenaire rods and a ladder with one rung built with the rods.

Analysis

All sessions were videotaped and prepared for storage as a part of the RBDIL video collection that is .hosted on an online video repository, the Video Mosaic Collaborative (http://www.videomosaic.org). The video sessions were first described, transcribed, and coded for critical events. (based on Powell, Francisco, and Maher, 2003). The videotapes were then analysed across all sessions to identify themes about Ariels' algebraic reasoning. The Ladder Problem was chosen to analyse for this paper because it represented both his early and later algebra knowledge and enables us to trace the details of how he builds his solution.

RESULTS

Event 1: Developing a composite function (7th Grade)

The first event occurs during the 7th grade problem-solving session. After introducing the problem, the researcher asks Ariel to find how many Cuisenaire rods are needed to build a ladder with ten rungs. After building ladders with varying number of rungs he offers the answer of 32 and is asked about how he solved the problem. He was then asked to write up his solution (Figure 2).

Ariel: I take half of the number if it has a half, I will multiply it by two and subtract two.

Researcher: So like for ten, how would you do that?

Ariel: I think a ladder of five, then count the rods, multiply that by two and subtract two.

Researcher: And subtract two. How would you do that for nine?

Ariel: For nine, I would do eight, then I would multiply the number by two and then I would subtract two and then I would add three.

As indicated in Figure 2, Ariel offered a composite function depending on whether the number of rungs in the ladder was odd or even. He justified his rule for four cases and was satisfied that it addressed the conditions of the problem for those cases.

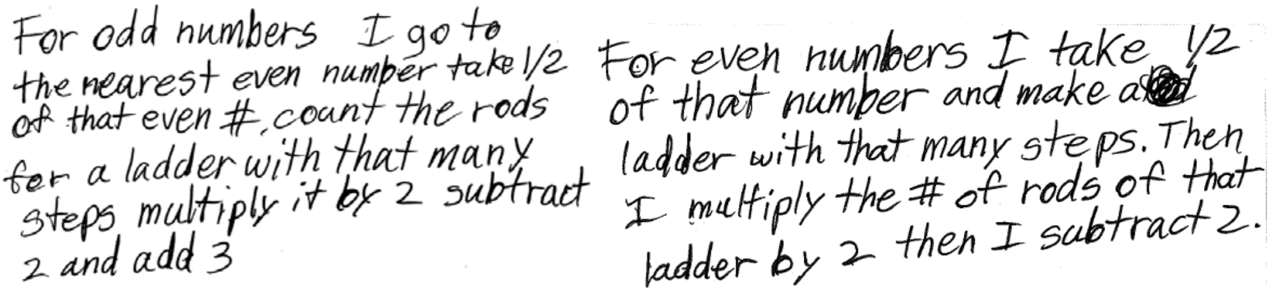

Figure 2: Student work - Ariel's rule for even and odd numbers.

Event 2: Using First Difference (8th Grade Interview)

Fifteen months later, an interview was conducted with Ariel in which he again was given the Ladders problem. When asked to determine how many rods are needed for a ten-rung ladder, he immediately built a table (Figure 3) and provided the following justification.

Ariel:	How many rods would you use to build a ten-rung ladder? Well, *I'm not going build it, I'll just do a X, Y table*. Um, that the difference between the Y, the Y, um, variables, was from 5 to 8 it increased by 3, from 8 to 11 it increased by 3...so you would just keep on increasing by 3, so until and then, it'll keep on going. *And then it shows that it's linear obviously because in a linear equation the first difference is always the same*. So, it's linear.

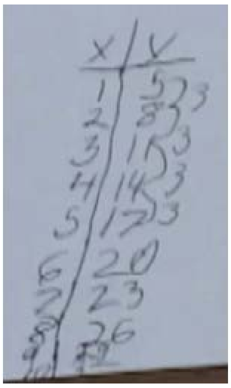

Figure 3: Student work - Ariel's X, Y function table.

During this session, although Ariel had available Cuisenaire rods to build his ladder, he chose not to use them and instead he constructed an X Y table as indicated in Figure 3 above. Notice that to the right of his Y column he noted the first order difference of 3 between each value.

Ariel completing his table, the researcher asked him how he could build a ladder with any number of rungs and Ariel *immediately* responds with his solution.

Ariel:	Um, how could you represent the number of rods needed to build a ladder with any number of rungs? That would be... hmmm... Oh! That's easy. Y equals 3X plus 2.

The researcher invited Ariel explain where his formula came from.

Interviewer:	Wait... where'd that come from?
Ariel:	Cause uh... how could you represent the number of rods needed to build a ladder with any number of rungs. So, you'd get the number of rungs from multiplying the, the...ladder which it is, like if it was the first ladder, second ladder, third ladder...multiply by 3, like on this one...it would be nine...and plus two is eleven. So, substitute the number, for, in each X, it would be 3 times 3 plus 2...it would be 9 plus 2... it's 11. And it works out, for every one.

The researcher then asked Ariel where the plus two came from, and Ariel responded.

Ariel:	Because... *I just looked at it* and if… if you multiply each by three… it's gonna be, um, m plus the y intercept, which is gonna be 2. *Cause if it's adding three each time, if you reverse this to when it was at 0, it would be a 2 right there.* Wait… yeah, it'd be a 2 right there. And then, *this [indicting 3] would be your slope of 3,* and *your y intercept of 2* [indicating the value of (0,2)]. And then it's a linear equation.

Ariel pointed out that his rule "works for every one". He made use of the first difference of 3 to identify the slope and noted that the y intercept was 2 to produce the rule of 3X+2 where X is the number of rungs in the ladder.

Event 3: Reflecting on past problem solving (8th Grade Interview)

After re-solving the problem the researcher invited Ariel to watch the video of his earlier work on the Ladder Problem in the 7^{th} grade and to compare how he solved it in 7^{th} grade to how he just solved it minutes earlier.

Interviewer: So now that you saw how you did it back in the day and how you did it now, what can you tell me about the two solutions? Like how can you compare them?

Ariel: Well, like I said before, *that way would be the long way* and then now it's like you just combine it together to make an equation all together. *Like, back then I didn't really know that much about equations, so it would just be like separate steps, and I didn't know, instead of combining them*. Like there, that's say, like you know how an equation could have the factored form and the expanded form. So this would be the expanded and that would be the factored. That would be the long way, *like all the different pieces all separate instead of just putting them into one.*

Ariel described his earlier work as "the long way", and indicated that he "didn't really know that much about equations". He also pointed out the efficiency of his new approach.

Interviewer: Oh, I see. So you're saying that you could take all these steps here and then just combine them?

Ariel: Yea, probably. But it would be more difficult to explain the part about going to an even number in the equation. Well, actually, it would be, cause if your x is going to equal nine then you just do, it'd be x minus one, you get the eight, which is eight. *But it wouldn't work every time*... cause then you would get an even number, wait, *no it would work almost every time*.

Ariel continues to explore other cases and concludes that the solution is basically a check.

Ariel: Say, well this *would be basically a check up*, in a way, to see if it matches that, if the formula could help you match the number of rungs to the number, I mean, the number of rungs to the number of pieces. So then, here you just take the number of rungs and multiply it and then you add two. So, yea they were both effective in a way, in their own little way. Yea, cause here it would be like a check, a good check right there.

Ariel maintained that both solutions were "effective", suggesting that he was open to more than one approach to establish the correctness of his reasoning.

DISCUSSION

Ariel began his exploration by identifying two rules to account for the number of rods required to build ladders of finite number of rungs. He partitioned his solution into two cases: odd and even. In so doing he built a composite function that provided a solution of ladders for certain numbers of rungs. It is interesting that Ariel first described his solution in words, using language that was familiar to him from his arithmetic learning. This approach contrasts starkly to his later problem solving in which he choses to construct a function table, indicate the first finite difference to identify the slope and justify the value of the y intercept to produce an elegant, general solution to the same problem. Confronted with his earlier work by viewing the video, he shows amusement in the video clip as he acknowledges the correctness of his earlier work, yet recognizing that he now produced a general solution in the form of a linear equation, whose components are connected to the function table he created. His use of the formal mathematics language, elegantly stated, shows his growth in connecting the meaning to the language to the symbolic notation. Ariel recognizes that both approaches have merit; they provide a check for him, connecting the informal with the formal. He has shown that the visual model created with the rods from his earlier problem solving was no longer needed. In his later work, he can imagine and represent his ideas with numbers and symbols, as well as with formal language. Ariel's case is important in that it underscores the need for providing opportunities for students to make use of their personal representations of the problem, using what is already known, and building on that knowledge.

Early, informal open-ended problem solving opportunities provide the ground work on which learners can build their knowledge. These investigations are not to be viewed as activities that should be given to students merely as a follow-up to procedural instruction; rather, these explorations are not only the beginning but the *essence*. Teachers must find the time in their classrooms to provide students with the opportunities to explore, examine, revisit and connect ideas and concepts through investigations that enable the learner to build strong intuitions of the problem conditions. Engagement in activities, such as the Ladder Problem, are needed to build a strong foundation for gaining insights and deeper understandings..Ariel had these opportunities and built his algebra knowledge on solid ground. His success is indicated in the elegance of the solution he provided, the understanding he had of his earlier work, and the confidence he had in offering his providing clear justifications for his work when asked..

ENDNOTE

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