PATTERN GENERALIZATION PROCESSING OF YOUNGER AND OLDER STUDENTS: SIMILARITIES AND DIFFERENCES*

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We compared the pattern generalization processing of US elementary and middle school students on similar tasks before and after a long-term exposure to a multiplicative-driven mathematics curriculum. Results show that both groups learned to express structural function-based generalizations as a result of their firm understanding of multiplicative relationships. However, the elementary group consistently processed inductively in nonsymbolic algebraic terms, while the middle school group processed the same tasks deductively in symbolic algebraic terms.

INTRODUCTION AND RESEARCH QUESTION

In this paper we resolve the following research question:

> How do we characterize the pattern generalization (PG) processing of US elementary and middle school students on similar tasks before and after a long-term exposure to a multiplicative-driven mathematics curriculum?

Patterning generally involves searching for mathematical regularities and structures. In this study, we further characterize individual students' *PG* in terms of their ability to mutually coordinate their visual and non/symbolic inferential abilities that enable them to *construct* and *justify* a plausible function-based algebraic structure (Rivera, 2013).

Following Duval (2002), we assume that visual is different from mere vision. While vision sees what it sees and experiences and needs physical action in order to completely apprehend an object, visual is neither physical nor mental but semiotic (pp. 320-322). Thus, a visual inferential skill toward patterns utilizes both epistemological and synoptical performance, where the epistemological aspect involves constructing an appropriate signifier (e.g. formula) for a given signified and its class (e.g. the known stages in a pattern) and the synoptical aspect involves analytically seeing them through a sequence of focusing actions producing relations and organization of relations between them (ibid). In PG we assume that a synoptic grasp of an interpreted structure involves both theoretical (i.e. abductive and/or deductive) and empirical (i.e. inductive) justifications.

Following Heeffer (2010), algebraically structures are either symbolic or nonsymbolic. Both require analytical methods in which case the unknown quantities in a pattern task are represented abstractly in some way (e.g. the use of variables). However, nonsymbolic algebraic structures remain at the level of notational representation, while symbolic algebraic structures begin with and proceed to systematically manipulate the corresponding abstract representation (pp. 88-89).

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RECENT RESULTS ON PATTERN GENERALIZATION

Overall our interest in PG as a research activity stems from our view that patterning activity democratizes students' access to structures, in general, and functions, in particular. Looking at the two tasks shown in Figure 1, both the figural pattern and the function problem require the same mathematical analysis despite appearing to have two different contexts. Both tasks ask students to obtain values for near generalizations (cases 9 and below), far generalizations (cases 10 and above), and a formula that works for any case and can conveniently predict future outcomes.

Figural Pattern

Stage 1 Stage 2 Stage 3 Stage 4

Assume the pattern resembles a growing L shape. How many squares would there be in stage 6? 8? 12? *N*?

Function Problem

The data in the table show the cost of renting a bicycle by the hour, including a deposit.

Hours (h)	Cost in $
1	3
2	5
3	7
4	9

How much does it cost to rent a bicycle for 6 hours? 8 hours? 12 hours? *h* hours?

Figure 1: Two Patterning Tasks that Require the Same Structural Generalization

Three additional factors informed our decision to coordinate students' multiplicative-driven understanding and their PG (see Rivera (2013) for a detailed synthesis). First, students' verbal descriptions of generalizations in past studies were consistently categorized as nonfunction-based. Second, they usually had difficulty identifying what stayed the same and what changed in their constructed patterns. That even if some students were able to state interesting properties, however, the descriptions were too difficult to be converted algebraically. Third, given the sequential manner in which patterns were oftentimes presented to students, recursively additive generalizations (i.e. Next = Current + Common Difference) were favored most of the time and that naïve induction approaches in cases of covariational-correspondence relations were most prevalent. Figures 2A and 2B illustrate the two "weak" pattern processing relative to the tasks shown in Figure 1. In the case of Figure 2A, research has shown that many students were unable to convert recursively additive expressions in function form. In the case of Figure 2B, research has indicated that many students were unable to justify naïve induction (or other pattern spotting) approaches despite their success in converting a structure in direct formula form. In multiplicative-driven responses on PG tasks, as exemplified in Figure 2C, students initially stipulate and justify an abduced (i.e. hypothesized) common unit that they then inductively test (i.e. verify) over several more cases and generalize to any stage in the pattern.

A. Recursively Additive Generalization	B. Naïve Induction Generalization	C. Multiplicative-Driven PG
1 3 2 5 + 2 3 7 + 2 4 9 + 2 "Add 2."	1 x 1 + 1 = 2. But I need 3. 1 x 2 + 1 = 3. Then 2 x 2 + 1 = 5. 3 x 2 + 1 = 7 and 4 x 2 + 1 = 9. I think the formula is S = n x 2 + 1.	The L-shape pattern has two growing legs and a corner square. So: Stage 1 has 1 corner square and 2 groups of 1 square. Stage 2 has 1 corner square and 2 groups of 2 squares. Stage 3 has 1 corner square and 2 groups of 3 squares. Stage 4 has 1 corner square and 2 groups of 4 squares. Stage *n* has 1 + 2*n* squares in all.

Figure 2: Three Generalization Processing Relative to the Figure 1 Task

EMERGING PG PROCESSING OF GRADE 3 STUDENTS

We resolve our Research Question in three sections. In this section we present Year 2 pre-post results drawn from a two-year longitudinal study that started when the students were in Grade 2. For four consecutive weeks the second grade class of 21 students formally learned about the concept of multiplication through representations that conveyed the set and array models. Skip counting by 2, 5, and 10 was also a significant part of this unit. The class consisted of 14 males and 7 females; the majority were of Hispanic origins. In third grade, they continued to learn multiplication involving larger numbers, committed to memory the multiplication table from 1 to 10, multiplied up to 4 digits by a single factor, and worked through arithmetical problems whose products maxed to 100000. Figural patterning activity was not a stipulated recommended activity in the CA mathematics standards for second and third graders at the time of the study. Hence, in second grade the students pursued patterns as an enrichment activity for two weeks toward the end of the school year after state testing. In third grade, they dealt with patterns only during the two clinical interviews, which were the sources of the pre-post results shown in Table 1. The study utilized a pre-post repeated measure design. A pre-post instrument involving 3 figural tasks, see Figure 3, was administered during the individual clinical interviews. Pre-interviews were conducted in early October 2010 and post-interviews took place in mid-April 2011. The interview protocol was as follows: Before the first pattern was presented, the interviewer asked each student to recall how he or she expressed multiplicative expressions by going over a series of 3 visual tasks (e.g. Counting Stars Task in Figure 3). The student was then shown a copy of the pattern stages and asked to reconstruct them either by using the blocks on the table or by drawing them on paper. He or she then responded to the questions listed in Figure 3.

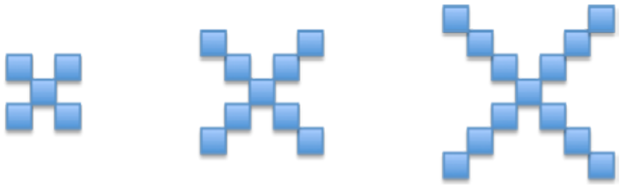
Cross Pattern (CP)	Two-Row Squares Pattern (TRP)
	
Flower Pattern (FP)	Counting Stars Task
	
1. Show me how stage 4 might appear to you. Use either the blocks on the table or draw it on paper. 2. Show me stage 5. Use either the blocks on the table or draw it on paper. 3. A friend asks you to describe your pattern. What do you say? 4. Can you describe stage 10 for me? Stage 15? Stage 100? 5. Will you describe your pattern in terms of groups?	

Figure 3 Tasks Used in the Grade 3 Pre-Post Assessment

For coding and analysis, verbal descriptions for each figural task were categorized along the following three levels: Additive and recursive (AR); Additive but composite-driven (Level I; AC); and Multiplicative (Level II; M). Numerical points were assigned, as follows: 1 point for AR; 2 points for AC; and 3 points for M. Totals were obtained for each student and then a paired t-test was conducted.

As indicated in Tables 1 and 2, extensive exposure to a multiplicative-driven mathematics curriculum had a significant positive effect on the third-grade students' ability to obtain direct formulas beyond additive recursive forms. The paired-samples t-test show that the scores were significantly higher on the posttest (M=7.3, SD=2.0) than on the pretest (M=4.6, SD=1.7; $t(18)=4.6$, $p < .001$, $d=1.5$).

EMERGING PG PROCESSING OF GRADE 8 STUDENTS

In this section we present two sets of data drawn from a three-year longitudinal study that started when the students were in Grade 6. For six consecutive weeks the sixth grade class of 29 students formally learned about figural PG using several units from the Mathematics in Context curriculum. While the students learned about integers and integer operations prior to learning figural PG, the number operations activity was implemented based on research implications then about the value of having students obtain a firm grasp of the arithmetical operations prior to any patterning activity. Also, patterning activity in sixth grade focused on the students' ability to visually infer structures on patterns since research implications then articulated the students' tendencies toward numerical- and recursively additive-based patterning at the expense of understanding and function-based formulas. The shift toward a multiplicative-thinking approach to PG took place when the students were already in eighth grade. In eighth grade multiplication was central to how the students learned Algebra 1 concepts.

	Pre	Post
Mean	4.58	7.32
SD	1.74	2.00
N	19	19
SEM	.40	.46

Statistically significant; mean of pre minus post equals -2.74; 95% CI of this difference – from -3.98 to -1.49

Table 1 Grade 2 Pre-Post Results

Pretest Frequencies

	CP	TRP	FP	%
AR	10	12	9	.54
AC	7	6	9	.39
M	2	1	1	.07

Posttest Frequencies

	CP	TRP	FP	%
AR	2	6	5	.23
AC	5	5	7	.30
M	12	8	7	.47

Table 2 Grade 2 Pre-Post Frequencies

The sixth grade students (mean age of 11 years; 12 males, 17 females; majority of Southeast Asian origins) participated in a pre-post assessment consisting of six tasks that were administered to them as homework problems over three consecutive days (2 tasks per day). The tasks were very similar to the ones shown in Figure 3, but they were simply asked to draw or explain. Figure 4 shows a representative sample pair (1 was figural, the other one was function-based). The design of the study utilized pre-post repeated measures. Each task was assigned the following points for formula construction: 0 for additively recursive expression or formula; and 1 point for an algebraically useful formula. The mode of justification for each task was also categorized separately as follows: recursive explanation (e.g. "I kept adding …"); and structural explanation (i.e. based on the context of the problem). Totals were obtained for each student and then a paired t-test was conducted.

1. Consider the growing ladder below.

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|-|   |-|   |-|
      |-|   |-|
            |-|
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a. Fill in the table.

X	1	2	3	4
Y	3	?	9	?

b. How many sticks are needed to form a 5-step ladder?

c. If a ladder has *n* steps, how many sticks are there altogether? Explain your answer.

2. The data below shows the cost, C, in dollars of painting a wall and the number of cans of paint, n, needed to paint the wall.

n	1	2	3
C	42	49	56

a. How much does it cost to paint the wall with 7 cans?

b. How about n cans? How do you know? Explain your answer.

Figure 4 Pre-Post PG Tasks Administered to Grade 6 Students

As indicated in Table 3, the paired-samples t test indicate that the scores were significantly higher on the posttest (M=4.9; SD=2.2) than on the pretest (M=1.9, SD=.8; $t(28)=7.2$, $p < .001$, $d = 1.9$), which meant that the students successfully expressed their structural generalizations in the posttest in algebraically useful terms. However, as shown in Table 4, the mode of justification remained numerical and very few employed structural understanding based on how a direct formula might make sense by inferring them on the stages. A recursive justification meant that a student simply added or subtracted until the correct value emerged, while a mere appearance match justification meant that a student simply substituted an input value to a constructed direct formula as a way of checking whether the formula was correct (e.g. "My direct formula for Figure 3 item 1 was y = 3x. I plugged in 2 for x and the result was 3 x 2 = 6, which had the same value of y on the table, so my formula made sense to me").

	Pre	Post
Mean	1.86	4.93
SD	.83	2.17
N	29	29
SEM	.15	.40

Statistically significant; mean of pre minus post equals -3.07; 95% CI of this difference – from -3.94 to -2.20

Table 3 Construction Pre-Post Results

Pretest Means		Posttest Means		
Recursive	Structural	Recursive	Mere Appearance Match	Structural
5.3	0.7	1.2	3.6	1.2

Table 4 Justification Pre-Post Results

In seventh grade, the students who continued to participate in the Year 2 study exhibited the same findings that we obtained on the posttest of the preceding year. Consequently, all their constructed direct formulas relative to linear figural PG took the constructive standard form y = mx + b that were mostly justified by a mere appearance match. In eighth grade, when the students acquired a multiplicative understanding of algebraic concepts, the nature of their PG forms also changed. Table 5 shows the percentages of correct PG by type over three years based on the post clinical interviews, which also indicates the different types of multiplicative-driven direct formulas that they generated. Figure 5 illustrates these types in relation to the Cross Pattern task shown in Figure 3. A deconstructive PG involves figural processing via decomposition with overlapping parts, which explains the converted formula consisting of multiplication and subtraction. An auxiliary-driven PG involves figurally seeing wholes first by adding auxiliary units (small squares), which explains the converted form involving the operations of multiplication and subtraction. A constructive nonstandard PG is an additively composite relation that simplifies into a constructive standard PG. A transformation-based PG involves the mereological process of reorganizing a figure into something that is familiar. In Table 5, we see that some PG processing types are not as frequently used as others.

PG PROCESSING OF GRADES 3 AND 8 STUDENTS

The results in the preceding two sections indicate similarities in the PG processing of younger and older students before and after a long-term exposure to a multiplicative-driven mathematics curriculum. Before formal instruction, their initial formulas were recursive in form and perhaps this is due to either their prior experiences or the sequential format of the tasks. Interestingly enough, both groups successfully transferred their understanding of multiplication to PG contexts that enabled them to construct and justify different types of direct formulas that were mostly constructive standard perhaps due to the limited nature of patterns that they dealt with (i.e. linear patterns). The transfer, in fact, occurred with very minimal formal training. Further, like their older counterparts, the younger group produced a significant number of additively composite (i.e. constructive nonstandard; early multiplicative) generalizations (see Table 2 pre-post frequencies).

Year	Recursive	Constructive Standard	Constructive Nonstandard	Deconstructive	Auxiliary-Based	Transformation-Based
Grade 6 (n=29)		100%				
Grade 7 (n = 8)		100%		100%		
Grade 8 (n = 14)		100%	36%	86%	21%	14%

Table 5 Percentages of Correct PG by Type Over 3 Years Based on the Post Interviews

Constructive Standard S = 1 + 4n 1 middle square plus 4 groups of n squares	*Constructive Nonstandard* S = 1 + n + n + n + n 1 middle square plus n squares on each of 4 sides	*Transformation-Based* Move the squares so that you see 4 rows of n plus an extra square, so S = 4n + 1.
Deconstructive S = 2(2n + 1) – 1 2 diagonal sides each of length 2n + 1 minus a middle square that has been counted twice	*Auxiliary Driven* $S = (2n+1)^2 - 4n^2$ Larger square is $(2n+1)^2$, take away the added small squares consisting of 4 groups of n^2	

Figure 5 Different Multiplicative-Based Structures for the CP in Figure 3

One remarkable difference between the two groups is the algebraic nature of their PG. Results of the post clinical interviews indicate that the third grade group consistently employed inductive generalizing across task, unlike the eighth grade group that consistently exhibited deductive-driven generalizing, which they already started to manifest, in fact, in sixth grade on the basis of the post clinical interviews. Below are two representative post-clinical interview responses from Rudy (Grade 3) and Tamara (Grade 6) in relation to the Cross Pattern task shown in Figure 3. Tamara in lines 1-3 below ignored the standard step-by-step questions and proceeded immediately to establish a variable-based generalization that she then used to deal with any near and far generalization tasks.

For stage 1 there's 5, then 9, 13, 17. Then 1, 2, 3, 4 [stage numbers]. So there's 4 in between. So 1 times 4 + 1 [for stage 1], 2 times 4 plus 1…. Yeah, it would be S = n times 4 plus 1.

Rudy (R) initially drew stages 4 and 5. He then claimed that "*the rule is adding 4*." When the interviewer (I) asked him to obtain an alternative "*expression involving* *multiplication and/or multiplication that does not involve adding 4*," he focused on stage 5 first and saw that it had "*5 on each side.*" When asked to further describe stages 10, 15, and 100, he said: "[For stage 15], it's 15 + 1, then 15 on [each remaining side]. … [For stage 100, it's] 100 x 4 + 1."

I: So what does that mean? What does each number mean?

R: This one [4] is for the sides, this one's [100] for the number on each side, plus the one square in the middle.

DISCUSSION AND CONCLUSION

Results of the two longitudinal studies conducted with the two groups of learners yield two insights that were not evident and clearly articulated in earlier reported studies on pattern thinking. *First*, we offered sufficient empirical evidence of the strength of coupling PG processing and multiplicative thinking together. Since multiplication as a concept fundamentally draws on common unit understanding (i.e. thinking in (equal) groups of some common unit), the same cognitive action applies to objects in patterning contexts. Across the two groups, we saw that students who held a firm understanding of multiplication overcame the gap between representational (figural) processing and (visual-to-algebraic) conversion. *Second*, due to the rather large samples and the longitudinal nature of the interventions, we also managed to capture stable PG actions over time that enabled us to infer similarities and differences in the two groups' generalizing performance on linear patterns. The differences, especially, should help us further characterize the nature of algebraic content that younger children are capable of exhibiting over time perhaps not in terms of early algebra versus something else but in terms of nonsymbolic and symbolic algebra. In closing we recommend that studies are necessary that can determine how early and what factors in elementary students' learning experiences might they be able to process structural generalizations in a deductive context.

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