THE NEED FOR PROOF IN GEOMETRY: A THEORETICAL INVESTIGATION THROUGH HUSSERL'S PHENOMENOLOGY

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The students' internal need for proof is at the crux of their learning advanced mathematics. In this essay we adopt a Husserlian perspective in order to read the sociocultural factors that lead to the genesis of geometrical proof in ancient Greece, with the purpose to investigate the contribution of Husserlian phenomenology in pedagogies aiming to foster the students appreciation for proof. We argue that the transcendental character of Husserl's phenomenology may contribute in a coherent framework for addressing the whole spectrum of the students' identified internal needs for proof (notably, structure), thus contributing in more effective pedagogies.

INTRODUCTION

The students' learning and thinking about mathematics have been discussed through phenomenological ideas stressing the importance of communication and active argumentation (Gravemeijer, 1994; Radford, 2003). Nevertheless, it appears that little explicit discussion has been made about the contribution of phenomenological ideas about the students' appreciation of and need for mathematical proof, which includes the coordination of logic and the axiomatic structure to investigate the validity of a statement. Such a view of proof does not exclude its various functions (for example, verification, communication, explanation, systemisation; Balacheff, 1991; de Villiers, 1990; Hanna, 2000). Recently, Zaslavsky, Nickerson, Stylianides, Kidron and Winicki-Landman (2012) addressed both mathematical and pedagogical perspectives about the need for proof, while Grabiner (2012) discussed the reasons that lead to proof from a historical perspective. In this theoretical essay, we consider historical, philosophical and empirical evidence to investigate the contribution of Husserlian ideas in this discussion. We argue that Husserl's transcendental phenomenology (Husserl, 2001) may contribute in an interpretative framework for gaining deeper understanding about the sociocultural factors that lead to the genesis of proof in geometry in ancient Greece and for understanding and pedagogically fostering the students' intellectual need for proving in geometry. Hence, we consider the fundamental question: *What is the contribution of Husserlian ideas in fostering the students' need for proof in geometry?*

HUSSERL'S TRASCEDENTAL PHENOMENOLOGY

Intentionality is central to Husserl's phenomenology, referring to "the conscious relationship we have to an object" and, thus, every "intending has its intended object" (Sokolowski, 2000, p. 8). The *natural attitude*, "our straightforward involvement of things and the world", is contrasted with the *phenomenological attitude*, "the reflective point of view from which we carry out philosophical analysis of the intentions exercised in the natural attitude and the objective correlates of these intentions" (Audi,

2013. In Lindmeier, A. M. & Heinze, A. (Eds.). *Proceedings of the 37th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, pp. 329-336. Kiel, Germany: PME.

1999, p. 405). The intersubjective experience of the communicated shared meaning is contrasted with transcendental subjectivity in which there is an awareness of a phenomenon outside our subjective perceptual experience. For Husserl, the "objective and the subjective are correlative, but never reducible to one another" (Audi, 1999, p. 405) and the phenomenologically described given object (*noema*) is not to be confused with the subjective activity (*noesis*). Transcendental subjectivity is viewed "as a possible communicative subjectivity [...] through possible intersubjective acts of consciousness, it encloses together into a possible allness a multiplicity of individual transcendental subjects" (Husserl, 1974, p. 31). His method of phenomenological reduction (*epoché*) includes *bracketing* out the natural attitude and moving towards a phenomenological attitude by investigating the sedimented intentional history of the object to "seek for its "constitutive origins" and to reproduce its "intentional genesis" (Klein, 1940, p. 150). During epoché, *explicit thinking* is activated (not just "passive, thoughtless repetition of words"; Audi, 1999, p. 406). Language (oral and written) is central to Husserl's phenomenology, constituting the means for the *objectification* of the subjective experiences. Husserl's idealities differ from platonic ideas in that new knowledge is intentionally subjectively constructed *once* in history and every subsequent knowing requires the reactivation of this objectification (Derrida, 1989).

PROOF AND PHENOMENOLOGY IN MATHEMATICS EDUCATION

The need for proof in mathematics education

Research suggests that the students lack a comprehensive understanding of the various functions of proof. For example, 14-15 years old students appeared to most commonly identify the verification function of proof, followed by explanation and communication (Healy & Hoyles, 2000), while the vast majority of younger students did not express a need for proving or specifically for 'proving the obvious' (Kunimune, Fujita & Jones, 2009; Williams; 1980). With respect to older students, first year advanced-level mathematics students appeared to obtain a theoretical understanding of the systematisation function of proof, but they were not convinced of its verification function, which may be linked with their reluctance to employ proof (Coe & Ruthven, 1994). Furthermore, in exam-type situations, second year mathematics undergraduates were found to consider the verification function of proof in their persuasion of others, without necessarily employing proof in ascertaining themselves (Moutsios-Rentzos & Simpson, 2011). Though the students' appreciation of proof widely varies depending on the identified by the students internal or external requirements (ibid), they need a reason to produce a proof (Balacheff, 1991). The external need for proof may be realised as an 'external conviction' proof scheme (Harel & Sowder, 1998) and/or may be embedded within the situation (Moutsios-Rentzos & Simpson, 2011). Considering internal needs, Zaslavsky et al (2012) identify: *certainty* (verification of the truth of a statement), *causality* (explanation of the reasons why a statement is true or not), *computation* (quantification of definitions, properties or relationships through algebraic symbolism), *communication* (formulation and formalisation in conveying ideas) and *structure* (logical re-organisation of knowledge). These needs may be viewed as an extension of and cultivated through everyday activities. For example, learning environments that promote inquiry appear to promote the students' internal needs for producing a proof (ibid). Nevertheless, we argue that the internal need for 'structure' is more demanding, especially when other internal needs have been met (note, for example, the aforementioned reluctance to prove the obvious).

Phenomenology in mathematics education

Considering phenomenological ideas in mathematics education research, it appears that mathematics educators seem to differ from Husserl's. For example, though Radford (2006) shares Husserl's view that objectification occurs within the semiotic system employed to signify an ideality, he argues that "transcendentalism [...] leads to an irresolvable tension between subject and object" (p. 39). Radford (2003) proposed a semiotic-cultural approach to the means of *desubjectification* (cf. objectification), stressing the subjective nature of the constructed through semiotic activities meaning. He discussed the sociocultural and the psychological aspects of algebraic thinking, delineating the semiotic means employed in the students' construction of meaning in algebraic activities. Though his work sheds light in the development of the students' algebraic thinking, it seems to lack explicit investigation about the students' internal need for proof. Moreover, the geometrical signs are of qualitative different nature, as "it is necessary to combine the use of at least two representation systems, one for verbal expression of properties or for numerical expression of magnitude and the other for visualization" (Duval, 2006, p. 108). We argue that a Husserlian phenomenological framework may help in addressing the students' internal needs for geometrical proof.

Drawing upon Freudenthal's didactical phenomenology, realistic mathematics focus on the students' needs to propose teachings in which the learners are guided by the teacher to actively organise a 'real' situation with mathematical tools, thus coming to the *guided reinvention* of a mathematical idea (Gravemeijer, 1994). By putting the learners in a 'real' for them problematic situation, they experience a 'real' need to successfully survive it. Through a process of *mathematisation* of the real-world situation and the teacher's guidance, the students actively re-invent important mathematical ideas as meaningful resources required for their lived reality. Though mathematisation may allow for the need of mathematically modelling and resolving a real situation and, subsequently, incorporating these mathematical ideas within the mathematical world (Freudenthal, 1983), it does not address the ways of incorporating this new knowledge; for example, we posit there is no 'real' internal need for a proof logically founded on already accepted as (or logically derived) true notions.

Consequently, it is crucial to identify pedagogies that explicitly address the students' internal need for producing a desubjectified, axiomatically and logically derived argument in order to be ascertained and/or to persuade. Should the 'institutionalisation' of the inductively and actively constructed by the students ideas be based solely on the teacher? Is mathematical proof only a linguistic shift to a symbolic, official language? To what extent semiotic changes and mediation may provide internal support to proof? In the following sections, we adopt a Husserlian perspective to read the historical genesis of proof in ancient Greece, attempting to address these questions.

THE TRANSCENDENTAL ARGUMENT IN ANCIENT GREECE

The reign of the argued thesis over the arguing subject

In this essay, we employ a Husserlian perspective in discussing the genesis of proof in ancient Greece. Reading history reveals a network of possible necessities, rather than a linear assortment of events, which requires the unfolding of the sedimented historical layers within which the mathematical notions were objectified. Though visual, empirical or measurement arguments existed in pre-Greek mathematics, specific factors worked in the ancient Greek city (*polis*) so that geometrical assertions were proved based on logic and accepted or proven to be true assertions, including the

> disagreement between older results, the desire to establish elementary first principles, the logical structure produced when problems are solved by reduction to simpler problems, the role of argument in Greek society, the central importance of philosophical argument in Greek thought, and the major contributions to mathematics resulting from using proof by contradiction (Grabiner, 2012, p. 152).

In the *polis*, the equality of the citizens allows their quantification (and objectification) within a democratic power system that a citizen corresponds to a vote. The ruling power is not a subjective quality, but it is the objectively measured sum of all the favourable votes. The qualitative unity of the subject fades out to an objective countable unit. By objectifying the power relationships, the city ensures the continuity and coherence of its structure regardless of the subjects who hold the various posts. Furthermore, the *polis* is characterised by the importance of the transparency of the most important events of the citizen's life. The private life that carries a significant weight is publicly shared to actually obtain its importance. The private-centeredness was generally frowned upon and the Athenian 'idiot' was the person who lacked the reasoning skills or the will to positively contribute in the public affairs. The citizen-subject is subjected to (*ypo-logos*) the oral communication of shared reasoned ideas (*logos* refers to both reasoning and oral speech in Greek). Vernant (1983) stresses that within this strong sense of belonging and the pursuit of a commonly shared reality, the personal identity was not lost. Heraclitus noted that "although *logos* is common to all, most people live as if they had a wisdom of their own". The *logos* is common in the sense that all private understandings and reasoning are in agreement with (*homo-logia*) with the public *logos*. Moreover, the Greek divine appeared to be strongly linked with human activities; for example, the arguer's power of convincing the many was subsequently attributed to the goddess Peitho (persuasion). Hence, we posit that the divine, the private and the public appeared to be strongly interconnected in the *polis*, thus constituting a transcendental with a strong anthropological character (rather than a non-human transcendental idealism). Within this framework the psychology of the self carries the transcendental references of the *logos* and becomes a multiplicity of higher mental internalised social relationships (cf Vygotsky, 1978). Moreover, Vygotsky (1978) notes that the close relationships of (external) social processes and (internal) psychological processes that in "their own private sphere, human beings retain the functions of social interaction" (p. 164). Stressing a Marxian perspective, Godelier (1977) investigates the circumstances and the reasons under which "a certain factor assume[s] the functions of relations of production" (p. 36), while Vernant (1975) identified the formation of the *polis* as the decisive event that allows *logos* to gradually become the utmost measure of power, replacing bloodline or even economic status with the power to convince the majority of the citizens. Thus, the matters of the common interest become an issue to be debated and to be ruled by the rhetoric of the speaker; *it is not important who the arguer is, but what the argument is*.

The genesis of proof in geometry

The idea of proof historically arose in ancient Greece (Katz, 2009).

> Logic lets us reason about things that are beyond experience and intuition [...] The Greek proofs by contradiction changed the way later mathematicians thought about the subject-matter of mathematics. Mathematics now had come to include objects whose existence cannot be visualised and which cannot be physically realised [...] Logical proof created these new objects [...] (Grabiner, 2012, p. 152)

Szabó (1978) claims that the study of incommensurable magnitudes and the irrational numbers and the notion of deductive proof did not meet any practical needs, but autonomous conceptual needs of a transcendental nature driven by the necessity of finding a shared reasoning. He studies the contemporary language of the market and the everyday life and discusses the ways that these words entered the mathematical language. He posits that proof was evolved as a shift of the word 'show': from making something literally visible to making "the truth (or falsity) of a mathematical statement visible in some [not necessarily visual] way" (p. 189). The movement towards showing the non-perceptual constitutes the quest for the commonly accepted *logos*, since the argument was not bounded from the subjective perception, but laid within a conceptual extension of the perceived reality, thus allowing the discussion of, for example, dimensionless objects (points) and unidimensional objects of infinite length (lines).

In addition, the quest for achieving the widest acceptance of an argument may have resulted in constructing geometry from the fewest commonly accepted truths and logic. The term 'common notion' may draw our attention to the common understanding of the humans' bodily experience. Hence, Euclidean Geometry may be viewed as the result of the ways that world perceptually appears to humans (visual, straight-looking lines). Common perceptually derived notions, objectified reasoning and quantified qualitative relationships may have been some of the elements that lead to the objectification of the mathematical argument through the notion of proof. It required many centuries before it reached its Husserlian ideal form, by challenging the a priori perceptually derived necessities within which geometrical ideas were built, thus allowing for the mathematical ideas to be constructed within whichever framework the mind chooses: axiomatic logically reigned worlds of infinite possibilities and choices. This allowed for mathematical counter-intuitive objects (such as the Weierstrass function) to be constructed/defined and their inescapable (within the chosen framework) consequences could be revealed through an anthropologically objective proof that transcends both the human bodily experience and its idealised extensions.

FOSTERING THE STUDENTS' NEED FOR GEOMETRICAL PROOF

In which way such a reading of history may inform a pedagogy aiming to foster the students' internal need for proof? The central phenomenological idea is summarised in going "back to 'the things themselves'" (Husserl, 2001, p. 168). From our perspective, this implies a pedagogical design within which the students' 'natural attitude' (for example, not to prove something obvious) can be suspended, allowing the possibility for the reactivation of the intentions that lead to the genesis geometrical proof.

We posit that at the crux of the students' appreciation of proof lies their acknowledging the existence of commonly accepted notions within which the geometrical objects are built. For example, these common notions may be related to the shared bodily perceptions and may be viewed as mental ideal extensions of the perceived world (cf. Lakoff & Núñez, 2000). The importance of the practical use of geometrical ideas may also strengthen the students' need for addressing a mathematical idea. In this way, the students may build a coherent framework deductively linking commonly accepted 'perceived' notions with the geometrical ideas. The need for proof may still not be evident, but we argue that by acknowledging that the geometrical ideas are corollaries of a priori notions, which, though related with human experience, are clearly beyond the reach of human perception, the students would be in the position to take the next step: to challenge these notions, to allow their mind to choose an alternative a priori framework within which reason can act. Acknowledging the a priori framework may internally justify the metacognitive processes of a 'what if ...' analysis to be applied. Thus, the need for proof derives from the metacognitive ability to identify and question the sedimented within language premises of the geometrical object. Harel's (cited in Zaslavsky et al., 2012) suggestion to teach 'neutral geometry' fits with this perspective, embodying a way of investigating alternative frameworks and their consequences. Within a mind game of choosing different a priori combinations, proof becomes the only way of accessing and evaluating the mathematical truth of the assertions, since the constituting framework is not perceptually derived, nor reachable. Language and symbolism allows the communication of those ideas, while reasoning and the chosen a priori are the bedrock upon which the truth of an assertion is proved. Hence, the students may extend their awareness of the transcendental as a conceptual non-perceptually bounded experience of the mind.

In this essay, we discussed the contributions of the phenomenological method in reading the sociocultural factors that lead to the genesis of geometrical proof, in gaining deeper understanding about the psychology of the proving subject, as well as in appropriately designing pedagogies aiming to foster the students' internal need for proof. Zaslavsky et al (2012) note that in school "proof establishes truth rather than validates assertions based on agreed axioms" (p. 219), masking the fact that "modern mathematicians adopt axioms or hypotheses without perceiving them as evident or absolutely true" (p. 218-219). We argued that the transcendental aspect of Husserl's ideas may contribute in a framework for addressing the internal need for 'structure' of proof. In phenomenological analyses "sedimented thought must be reactivated and its meanings revived" (Audi, 1999, p. 406). 'Going back to the things themselves' allows the metacognitive processes to challenge not only the solution of a given problem, but the very fabric upon which the problem is posed and solved. We posit that Husserl's phenomenology allows for the identification of the anthropological character of the complex processes that necessitate employing proof in learning mathematics, as well as their meaningful convergence within a coherent framework, incorporating semiotic structures, sociocultural conventions and subjective understandings.

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