MEDIATING THE MEANING OF ALGEBRAIC EQUIVALENCE: EXPLOITING THE GRAPH POTENTIAL

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Within the Theory of Semiotic Mediation, this research study investigates how the Graph representation provided by the Aplusix CAS can become a tool of semiotic mediation to make the meaning of algebraic equivalence emerge. The analysis of the potentialities of the Graph and the design of specific tasks aiming at exploiting them, are presented. More precisely, it is investigated how this representation may contribute to develop the meaning of equivalence as a structural relation which condensates the procedural processes underlying the transformation of algebraic expressions. A teaching experiment is described and some results are discussed.

INTRODUCTION

This contribution aims at exploiting the potentialities of a specific component of the Aplusix tool (Nicaud et al., 2004), namely the representation of algebraic equivalence by means of a graph. The 'Graph representation' in Aplusix is a particular type of representation of a set of algebraic expressions which become the nodes of the graph. A relevant body of research which concerns the issue of equivalence between algebraic expressions is focused on the difficulties encountered by students in understanding algebraic equivalence as well as in dealing with manipulation of various forms of algebraic expressions (e.g. Ball et al., 2003; Kieran, 1984). As far as symbolic manipulation is concerned, the issue of addressing the dichotomy between syntax (rules stating how to deal with symbols) and semantic (meaning of equivalence) arises. This distinction can constitute a cognitive obstacle to deal with algebraic manipulation. In fact, as argued by Booth (1989)

> "our ability to manipulate algebraic symbols successfully requires that we first understand the structural properties of mathematical operations and relations[..]. These structural properties constitute the semantic aspects of algebra" (pp. 57-58).

A valuable contribution to the research in this area comes from studies involving the use of Computer Algebra Systems (CAS) devoted to foster the equivalence meaning (e.g. Kieran & Drijvers, 2006; Lagrange, 2000; Artigue, 2002). Artigue underlines how the potential for these tools, compared with paper and pencil, consists mainly in giving the opportunity to make students face tasks which, at the same time, can foster both syntactic and semantic aspects.

Elaborating on these studies, our research is centred on the idea that symbolic manipulation can be conceived as a condensation of procedural relations – transformation of algebraic expressions according to rules respecting the equivalence – and structural relations – which are established between expressions obtained by procedural manipulation. According to Booth (1989), this makes syntactic and

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semantic aspects of algebraic manipulation be combined. We formulated the hypothesis that the Aplusix CAS, and in particular the use of the Graph representation to carry out algebraic manipulation, allows one to refer to the mathematical meaning of equivalence class of algebraic expressions.

THE GRAPH IN THE APLUSIX CAS

Aplusix is a CAS in which the user can perform both numerical and algebraic calculations. The computational objects of the CAS consist of algebraic expressions; each of these is enclosed in a box which constitutes a node of the Graph (Fig. 1, Fig. 2). The treatment of an expression is carried out within a box; new boxes can be added to the Graph. Two settings are available for creating a new box: the student can choose to create an empty box or to create an identical box to the previous one.

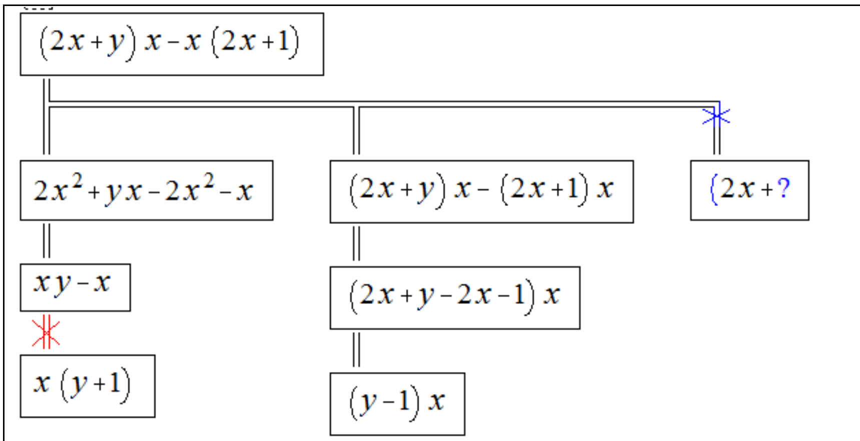

Figure 1. Black (red-crossed) lines indicate that two linked expressions are (not) equivalent, blue-crossed lines that the expression is not well-formed.

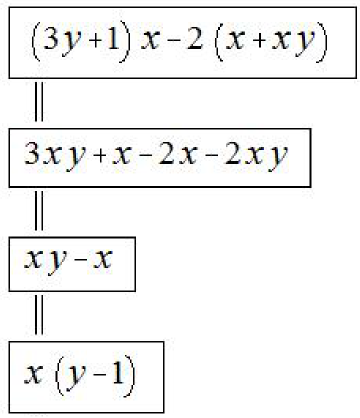

Figure 2. A graph vertically developed.

Each box enclosed in the Graph can be linked by three different signs to both the box from which it has been obtained and to another box, in case another one originates from it. In fact, when students carry out a calculation, they can exploit the feedback provided by these signs. Three different 'feedback-signs' signal whether the equivalence between the expressions filling two consecutive boxes has been respected or not. Black lines indicate that the equivalence has been respected, blue crossed lines indicate that one of the two expression is not syntactically well-formed (e.g. a parenthesis is missing), red crossed lines indicate that the two linked expressions are not equivalent (Fig. 1). Each box containing an expression can be the root for one, or more than one other boxes, in which other expressions can be represented. In other words, the Graph representation in Aplusix, which is two-dimensional, offers a very peculiar way of expressing equivalence between algebraic expressions (Fig. 1). As a consequence, the Aplusix CAS overcomes the mere top-down flow from the given expression to the required expression which is usually carried out by means of a single chain (starting with the given expression and ending with the required expression).

Actually, the iteration of generating new boxes containing equivalent expressions produces a web of expressions, that is a set of expression one related to the other by a link representing the equivalence between them; in such a web the order in which one expression is obtained from another disappears, whilst what becomes evident is the equivalence relation linking the different expressions.

THE THEORY OF SEMIOTIC MEDIATION

The Theory of Semiotic Mediation (Bartolini Bussi & Mariotti, 2008) provides us with theoretical tools to frame the analysis of the semiotic potential of the Graph representation in mediating the meaning of algebraic equivalence and to plan the didactical intervention based on its use. According to this theory, the Graph as it is implemented in the Aplusix CAS, can be considered as an artefact. The use of it, in order to carry out specific tasks, make students construct personal meanings which are deeply related to the actual use of it. Then, the role of the teacher consists in making students' personal meanings gradually evolve to mathematical meanings. An artefact, when intentionally used by the teacher in promoting this process, becomes a tool of semiotic mediation (Bartolini Bussi & Mariotti, 2008, p. 754). The semiotic potential of an artefact emerges through the twofold relation that such an artefact has both with the meanings emerging from its use to accomplish a specific task, and with the mathematical meanings which are evoked in such a use, as they are recognized by an expert (Bartolini Bussi & Mariotti, 2008, p. 754).

Planned didactical activities, which aim at making meanings emerge, constitute the basis on which the teacher orchestrates the evolution from personal meanings to the mathematical meanings. In the Theory of Semiotic Mediation, the organization of a teaching/learning sequence is based on didactic cycles (Bartolini Bussi & Mariotti, 2008, p. 754). Each didactic cycle pursues a specific didactic goal by means of a sequence of different activities aimed at developing personal meanings (solving tasks using the artefact), producing individual signs (writing reports related to the task solution), making students' personal meaning evolving towards mathematical meaning (discussing under the guidance of the teacher).

THE SEMIOTIC POTENTIAL OF THE GRAPH REPRESENTATION

The leading principle of the research study is that the Graph representation in the Aplusix CAS can be considered an artefact fostering the meaning of equivalence between algebraic expressions. As explained above, any graph can develop in two directions (Fig. 1, Fig. 2): vertically (creating box containing expressions one under the other) and horizontally (creating boxes of expressions which are one next to the other, starting from the same expression). The final product of this representation refers to an equivalence class of expressions; thus, we can refer to it as Equivalence Graph. Aplusix manages potentially infinite webs of expressions. The navigation of very large graphs is facilitated by the use of the vertical/horizontal slide bar. The possibility of representing algebraic expression through a graph permits a new approach to symbolic manipulation and, in particular, to the meaning of equivalence among expressions. Producing an Equivalence Graph means transforming a given expression into equivalent forms which can be transformed themselves into equivalent forms and so on. The iteration of this procedure makes emerge the symbolic manipulation as a condensation of procedural relations into a structural relation. In fact, the Graph can be considered as a representation of the structural relations between algebraic expressions that, at the same time, remain linked one another through the transformation procedures that have been carried out to generate each of them. The artefact, thanks to both its structure and the ways to act on it, allows the user to break the rigid *direction of the manipulation* towards either an expanded form (canonical form) or a factorized form (if it exists) of an expression. In this way, not only can the Equivalence Graph mediate the meaning of equivalence between expressions, but also the produced web of equivalent expressions makes it possible to refer to a complex system of expressions as differently related through symbolic transformation, and structurally related through the equivalence relation.

THE TEACHING/LEARNING SCENARIO

A teaching experiment was designed with the aim of exploiting the semiotic potential of the Graph representation in Aplusix for developing the mathematical meaning of algebraic equivalence, as well as the meaning of algebraic equivalence between transformed expressions. The implementation of the teaching/learning experiment involved two 9th grade Italian classes in the second semester of the school year.

The students were familiar with the Graph that had been already used to perform calculation tasks with numerical expressions. They were also familiar with the feedback-signs, and their meanings had already been shared, as well as the meaning of algebraic equivalence. Then, in the introduction to the algebraic calculation, the same artefact, that is the Graph, is exploited to carry out manipulation of literal expression. The teaching sequence is composed by some didactical cycles; we will consider only a few of them. The first task consists in asking students to state if a set of three expressions contains expressions which are equivalent and, in the affirmative case, to prove the stated equivalence. Two different approaches to the solution are possible, each of them refers either to the structural or to the procedural meaning of symbolic calculation. Equivalence can be stated by means of creating a graph containing the whole set of expressions and checking whether or not the lines linking the three expressions are black (indicating the equivalence) or red (indicating the non-equivalence). Otherwise, it is possible to start a transformation process producing a graph per each expression with the aim of transforming one expression into the other.

The analysis of the graphs may lead students to realize that if two of these share at least one expression they can be considered as part of the same larger graph. The comparison of students' solutions in the collective discussion will be aimed at developing the mathematical meaning of equivalence class of algebraic expressions: any graph will become an Equivalence Graph. Building on this, new activities will be proposed with the aim of introducing a classification with respect to the form of the expressions so that, finally, identifying the canonical form to be selected as a specific representative of the equivalence class.

CONDENSING PROCEDURE AND STRUCTURE INTO THE GRAPH REPRESENTATION

We are going to analyse some excerpts from the classroom discussion after the first activity concerning the comparison of three expressions. Objective of the discussion is fostering the emergence of the mathematical meaning of equivalence class, elaborating on the personal meanings emerging from the use of the Graph and, in particular, on the idea of two-dimensional web of equivalent expressions. The teacher opens the discussion coming back to the task by asking students how they have solved it. The teacher shows to the classroom files in Aplusix containing the solution given by students; she also uses the CAS to transform some expressions given in the task so as to give evidence of specific issues emerging during the discussion.

Marco: We made three schemes and in the B scheme the third expression in the second column (Fig. 1) is identical to the fourth expression in the C scheme (Fig. 2). We also copy this latter expression in the B scheme and everything was OK.

Teacher: Is that enough to prove the equivalence?

Martina: Every expression of the B scheme has to be equivalent to every expression to the C scheme.

Teacher: So, do we have to check any expression of a graph with any expression of another graph to say that the two expressions are equivalent?

Chorus: No.

The teacher tries to move from a local approach, in which the equivalence is respected if two (or more) graphs contain two identical expressions, to a more global one where the sufficient condition to state that two expressions are equivalent consists in proving the equivalence between two specific expressions: one belonging to a graph, the other one to another graph.

Teacher: So if I see the presence of at least two equivalent expressions, one belonging to one graph, the other to the other graph, what can I do?

Matteo: A new graph.

Teacher: A new graph […] Do you agree with him? How can I do?

Matteo: You can attach the second graph to the first graph by linking by chance one of the expressions of the first graph to one of the expressions of the second graph. In this way, you obtain a single graph containing the two graphs.

Teacher: Well, is there something which characterizes the expressions of the new graph?

Chorus: Are equivalent!

After having expanded a single graph into a new bigger one, by connecting two graphs (within the CAS environment), the teacher tries to move a step forward selecting a single expression which can stand for the whole graph.

Teacher: The procedure of expanding graphs by linking two or more of them, may suggest us the idea that is possible also to reduce a big graph to a smaller one, do you agree?

Chorus: Yes.

Lorenzo: Yes, we could also compact it, reducing it to a smaller one.

Teacher: How small?

[…]

Lorenzo: Since all the expressions which the graph is made of are equivalent, the graph can be compacted into a single expression.

[…]

Teacher: Well, since we all agree that it is possible to reduce a graph to a single expression, is there any idea of how it could be convenient to choose this expression?

Emma: I would choose the less complicated expression.

The discussion gives clear evidence of how the teacher uses the Graph representation as a tool of semiotic mediation in respect to the equivalence relation, as it is established either by expanding a graph or by compacting two graphs. Focusing on any pair of expressions belonging to the same graph it is possible to state the equivalence between them; at the same time a sequence of connected expressions may represent the procedure used to transform the former into the latter. In other words, the structural meaning of equivalence maintains memory of the procedural meaning.

The teaching sequence goes on with the aim of introducing the idea of representative of an equivalence class. The introduction of specific forms of expressions - canonical (or expanded) and factorized, if it exists - is made through classification tasks in which students are asked to recognize in a graph which are the expressions of a particular form (Maffei & Mariotti, 2011). Then, in the subsequent didactical cycle, students are asked to produce themselves expressions having a required form.

THE GRAPH REPRESENTATION AS AN INTERNALIZED TOOL

Interesting evidences of the role played by Graph representation in the construction of the meaning of equivalence comes from the results of a delayed test passed two months after the end of the teaching sequence. The test is performed in paper and pencil; in the written productions traces of the use of the Aplusix Graph representation clearly appear: students draw a graph (sometimes quite large) representing the web of equivalences where to select the expression(s) they need. Moreover, the solutions reveal how students has appropriated to this tool, transforming it in a personal mode of representation to be exploited in treating algebraic expressions, specifically to obtain particular forms and relate them in the equivalence class. In a vygotskian perspective, the Graph has been internalized (Vygotsky, 1978).

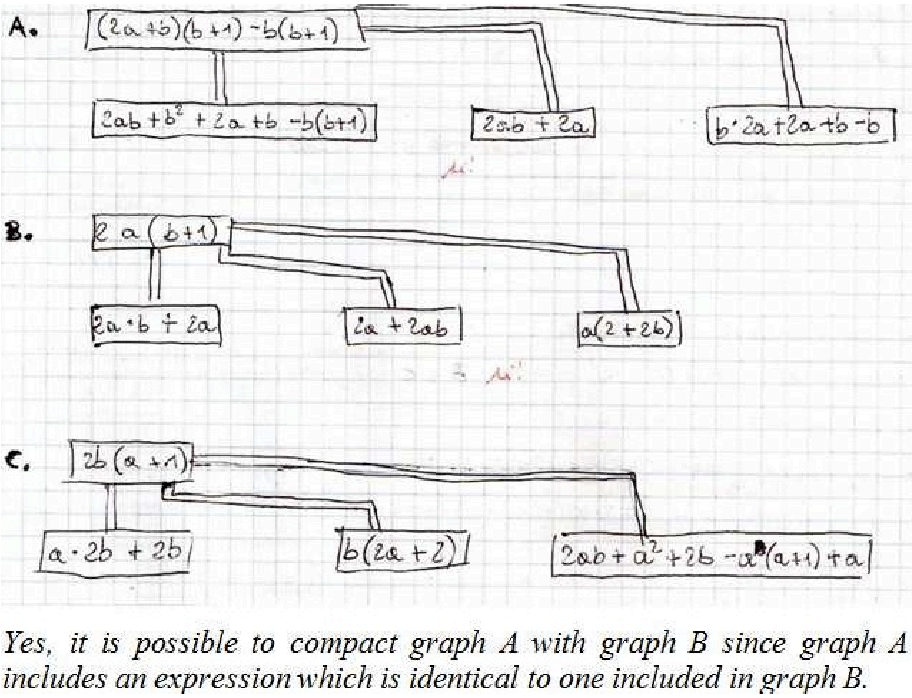

Figure 3. Internalization of the Graph representation.

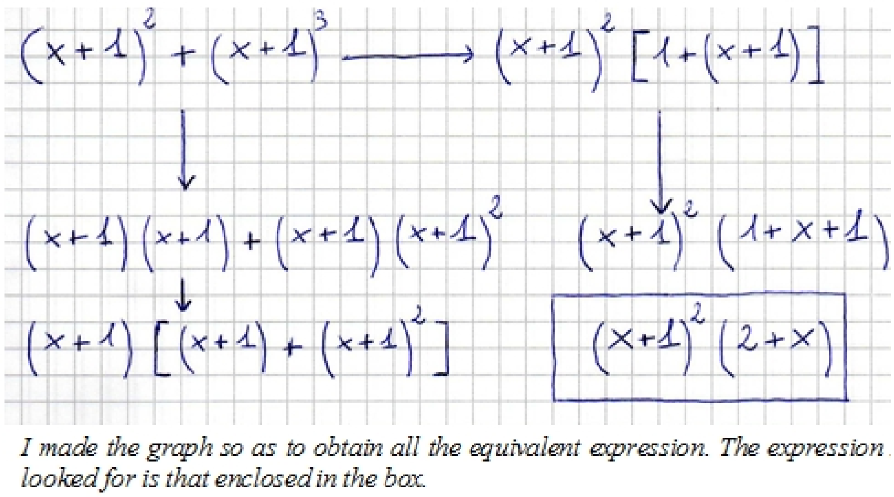

Figure 4. Obtaining the factorized form of an algebraic expression in a graph in paper and pencil.

Fig. 3 shows the solution given by Francesco to a task asking to state if some of the three given expressions are equivalent. The student reproduces three Equivalence Graphs – one per each given expression - as if he worked in the Aplusix environment. Then he proves the equivalence recognizing that two expressions belonging to two different graphs are identical.

The internalization of the Graph representation as a tool to think about equivalence and equivalent forms of an expression emerges with more emphasis in the task asking to produce specific forms of an expression.

When solving a task asking to produce the factorized form of an expression, Giacomo (Fig. 4) carries out different treatments of the given expression and produces an Equivalence Graph representing them, the solution is identified within such a graph. In general, the analysed protocols show how the Graph representation has become a tool suitable to refer to both the procedure (making vertical chains of equivalence possible) and the structure of the equivalence relation (embedding the vertical/horizontal chain in a single object).

CONCLUSIONS

The last protocols, referring to the delayed test, witness the students' appropriation of a way of representing algebraic expression, which clearly comes from the Aplusix Graph. Different vertical/horizontal chains, representing the transformations of an expression into another, are finally embedded in a single object, representing a class of equivalence, within which different forms of expressions can be identified to solve the task. According to the Theory of Semiotic Mediation, the designed teaching sequence has combined activities with the artefact and mathematical discussions to make personal meanings evolve into mathematical meanings: the Graph, thanks to its potential to referring to both the procedural meaning and the structural meaning of the equivalence relation, has become a mathematical tool of representation.

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