CONSTRUCTING THE FUNDAMENTAL THEOREM OF CALCULUS

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This research report is the fourth one in a series about learning integral calculus in high school. In designing a curriculum that supports an improved cognitive base for a flexible proceptual understanding of integration, we have chosen to use accumulation as core idea. We defined three conceptual milestones: Approximation, Accumulation Function and the Fundamental Theorem of Calculus (FTC). In this paper, we focus on the following questions: What is the structure of the FTC in terms of elements of knowledge? What operational definitions for these elements allow the researcher to follow students' construction of the FTC? How does the process of knowledge construction occur during an instructional intervention?

INTRODUCTION

Students have serious problems with understanding the concept of integral (e.g., Thompson, Byerley & Hatfield, in press). Many students have a tendency to see integral calculus as a series of procedures with associated algorithms and do not develop a conceptual grasp that could give them the desirable versatility of thought (e.g., Thompson, 1994). Thus, instead of a proceptual view, they have, at best, a process-oriented view. This may be due in part to a lack of opportunity to experience these processes directly. In order to develop an improved cognitive base for a flexible proceptual understanding (Gray & Tall, 1994) of the integral and integration, it has been proposed to use accumulation as core idea (Thompson, 1994; Kouropatov & Dreyfus, 2009). With respect to undergraduates, Thompson and Silverman (2008) "believe that understanding accumulation … can be part of a coherent calculus that focuses on having students see connections among rate of change of quantities, accumulation of quantities, …" (p. 13). Our claim is that with minor changes this is relevant for high school students as well. We designed a curriculum (Kouropatov & Dreyfus, 2012a) supporting students in constructing integration as a conceptual aggregate of knowledge elements from approximation via accumulation to the FTC. The adopted research methodology allowed us to closely observe students' knowledge-constructing processes of approximation (Kouropatov & Dreyfus, 2011) and accumulation function (Kouropatov & Dreyfus, 2012b). Here, we report on teaching episode where students deal, for the first time and in an intuitive manner, with the FTC. Hence, this paper focuses on the questions formulated in the abstract.

FTC: PRELIMINARY CONSIDERATIONS

Following Newton and Leibniz, the idea of the integral developed in two directions: integral as a limit of a certain sum (definite) and integral as antiderivative (indefinite). The notion of definite integral became an important tool by shedding light on many problems in science. The notion of indefinite integral led to a development in analysis

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that is (or at least was until technology put numerical methods at the forefront of calculus) the core of differential equations. The understanding of the connection between these two notions should in our opinion be a main aim of teaching integral calculus. A common approach presents the indefinite integral as a formal "undoing" in the "table of derivatives", and then uses it to compute areas (definite integral). Then it stops. Students never learn why they use antiderivatives to compute areas. The connection is never established. So the question arises: Can integration be taught at the high school level in such a way that the connection is established? Our proposed answer to this is accumulation, which leads simultaneously to the indefinite as well as the definite integral. In our curriculum, students study the FTC after constructing the concept of the accumulation function of a given function (the definite integral with a variable upper bound). In this approach, the FTC says that the rate of change of the accumulation function of a given function is the given function itself.

KNOWLEDGE CONSTRUCTING AND ABSTRACTION IN CONTEXT

The main aim of this paper is to analyze how students construct the FTC as a result of our instructional intervention. We have adopted Abstraction in Context (AiC) (Schwarz, Dreyfus and Hershkowitz, 2009) as theoretical framework. AiC takes abstraction to be an activity of vertical reorganization of previous mathematical constructs in order to arrive at a new (to the learner) construct. The activity is interpreted in terms of epistemic actions performed by the learners for a specific purpose, in a particular context. The context includes the social setting as well as the learner's personal background, including previous mathematical constructs resulting from previous abstractions. Reorganization includes establishing new connections between such constructs, making generalizations, and discovering new strategies for solving problems. "Vertical" implies building a new level of abstraction over a previous level. An essential component of AiC is a model of three epistemic actions for describing and analyzing at the micro-level the knowledge constructing process:

- R The learner *recognizes* a previous mathematical construct as relevant in the present situation.
- B The learner *builds-with* the recognized constructs to achieve a local goal such as solving a problem or justifying a claim.
- C The learner uses B-actions to assemble and integrate previous constructs so that a new (to the learner) *construct emerges* by vertical mathematization.

In processes of abstraction, R-actions are nested in B-actions, and R- and B-actions are nested in C-actions. These particular epistemic actions have been chosen because they seemed to be relevant for processes of abstraction as well as observable. This working hypothesis has been effective in studies taking place in a large variety of contexts (Schwarz et al., 2009, and references therein).

DESCRIPTION OF THE RESEARCH

We designed a ten-session unit and implemented it with five small groups of advanced-level mathematics high school student volunteers. The unit was independent of what the students were learning at school. Here, we report on an activity from the seventh session of a group of two female students (A and B). We worked with groups rather than single students in order to make the knowledge constructing process more observable through their discussions. We decided to analyze the knowledge construction of the pair rather than that of each of the students separately. Our data include audio recordings, transcripts and session protocols.

THE ELEMENTS OF KNOWLEDGE AND OPERATIONAL DEFINITIONS

AiC requires an a priori analysis of the tasks proposed to the students in terms of the intended knowledge elements, their constituents, and links between the constituents. Based on theoretical considerations, including that the FTC is a hierarchical, nested concept with a proceptual nature, and didactical considerations, including rate of change of the accumulation function of a given function as a complex co-variational process of change (e.g., the rate of change of the area accumulating beneath the graph of a positive function while the "right border is moving"), we focus on the following knowledge elements:

- CAF "Change of Accumulation Function": Changing the independent variable causes changes in the dependent value (of the given function) and in the value of its accumulation function simultaneously.
- iCAF "Infinitesimal Change of Accumulation Function": for any continuous given function, the change of its accumulation function, when the change of the independent variable is very small, can be approximated as an appropriate term of accumulation.
- RCAF "Rate of Change of Accumulation Function".
- iRCAF "Infinitesimal Rate of Change of Accumulation Function": for any continuous given function, the rate of change of its accumulation function at a certain point can be approximated as the ratio between the iCAF and the change of the independent variable.
- FTC For any continuous function defined on some closed interval, the derivative of its accumulation function is the given function itself.

Operational definitions of these knowledge elements have been used to assess whether students have constructed the knowledge element. For example,

- RCAF We will say that students have constructed RCAF if they explicitly (verbally and/or graphically) calculate/represent the rate of change of the accumulation function as a ratio between the difference of the values of the accumulation function and the corresponding difference of values of the independent variable.
- FTC We will say that students have constructed the FTC if they express (verbally and/or graphically) that the derivative of the accumulation function of a given function is the given function itself for any value in the domain of the given function, and explain it using iRCAF.

Students' previous constructs are likely to be relevant during the activity; they include constructs for 'function' or 'variable', as well as others, which we assume to have been constructed in preceding sessions of the unit, for example:

RM "Rectangles Method": In a rectilinear coordinate system, the accumulating value of any continuous positive function can be approximated as a sum of terms of accumulation that are areas of rectangles.

CC "Complex Co-variation": When considering accumulation, the value of the accumulation depends on the function that accumulates as well as on where we stop accumulating (upper limit).

AF_M "Accumulation Function Meaning": For every upper limit there is a unique value of the accumulation that depends only on the function that accumulates and on the value of the upper limit. The Accumulation Function associates the value of the accumulation to the upper limit.

AF_P "Accumulation Function Property": The properties of the accumulation function (e.g. increase) follow from properties of the given function *f*.

RoC "Rate of Change" of a function is the rate at which the dependent variable changes with the independent variable.

DRoC "Derivative as Rate of Change": The derivative of a function at a certain point can be defined as the limit of the RoC of the function as the difference of the independent variable values approaches zero. The limit can be understood intuitively as a number that the RoC approaches when the change in the independent value approaches zero at that point.

FINDINGS

During six previous sessions the students had learned about and succeeded to explain the meaning of the CC, AF_M, and AF_P. As an introduction to the seventh session, they were asked to consider the accumulation functions of constant functions like $f(x)=2$, $g(x)=-0.36$ and linear functions like $f(x)=2x$, $g(x)=-0.5x$ on an interval like $[0,b]$. They succeeded to explain the properties of and to produce the algebraic representation of the accumulation functions (A and B denote the students, R the researcher):

129 R: Look at your answer for task b. What do you think is the connection between *Af* [the accumulation function] of *x* and *f* of *x*?

131 B: According to the rate of change of the function the accumulation grew, like, the accumulation was moderate…

132 R: Okay, then what... visually, what is the connection with this to this? Between this function and that function? [R points to the algebraic expressions, which were written by students.] What's the connection between the function...

133 A: *f x* is used as the derivative of that function

136 B: What? What's the relationship? That the accumulation equals the derivative... OK, I got it…

138 A: As we saw in the previous tasks *Af x* prime equals *f x*, when the *Af* of *x* is the accumulation function of *f x*. In these tasks we dealt with certain functions. Now we should check if this relationship exists in each function.

The students recognized CC, AF_M, AF_P as relevant in the present context. When they were asked to connect between *f* and its accumulation function, it seems that they established the connection (133) based on the specific algebraic expressions of the example. They conjectured a generalization and expressed a need to justify it (138). The next task asked them to analyze (using guiding questions) the illustration in Figure 1 (the horizontal segment starting at C and the more or less vertical scribbles in the light grey area were not present at this stage but were added later).

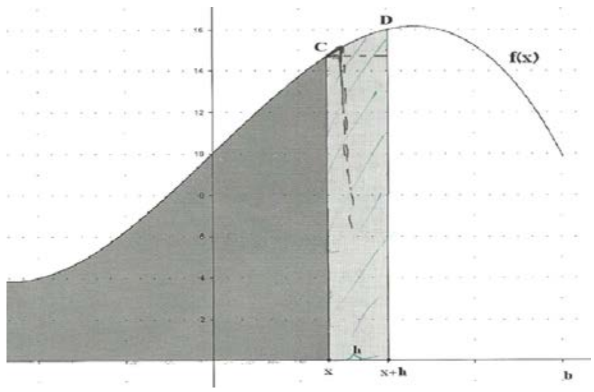

Figure 1

190 R: What is the change of the function of accumulation *Af* of *x* between *x* and *x* plus *h*?

191 B: It adds up like, what does it mean, what's the change?

192 R: How, you see the change of the function, what is it?

193 B: Yes, one more area

194 A: One more area

196 B: Which area?

197 A+B: This one [point on the light grey area in Figure 1]

From this excerpt we infer that the students are able to refer in parallel to three different entities: the independent variable, *f* as represented by the graph, and *Af*, which is not represented at all. This short episode allows us to point to the following nested epistemic actions: recognizing their previous constructs for CC, AF_M and building-with them to reach a new one construct – CAF. In the next task the students were asked to refer to the rate of change of *Af*:

200 R: Okay. What is rate of change, for now? Rate of change?

201 B: It's the size... another segment that you're adding to it, if it's big or small then like...

203 B: Slope, no?

205 A: Delta *y* over...

In 200-205, we can see some signs that fit with recognizing RoC but there is no connection to the task context. We can't identify any building-with here. It seems that including RoC as additional player in the plot is beyond the students' current grasp. But an explicit hint by the researcher brings immediate results:

211 R: No, I want the accumulation function. Accumulation

212 B: Aaah… *Af* of *x* plus *h*

214 A: Minus *Af* of *x* divided by *x* plus *h* minus *x*...

$$\frac{Af(x+h)-Af(x)}{x+h-x}$$

The students' observable behavior precisely fits the operational definition of RCAF. It appears that their experience from preceding lessons allowed them (with a little help) to construct RCAF in the context of the given task. In the next task the students were asked to refer (in the same context) to a very small change of the independent variable. They succeeded to connect the task with their previous knowledge:

219 A: As long as *h* is smaller

220 B: We can use a rectangle… that almost this area [draws the horizontal segment starting at C in the illustration, see Figure 1]

Here the students recognize as relevant their previous constructs for RM, CC, and CAF. They then start building-with these constructs:

229 R: If *h* is very small, what can be said about the difference between the area of the rectangle and …

230 B: Which rectangle are we talking about here?

231 A: Here

233 B: And the area that represents the change of the accumulation function

235 B: All of this, all up the top

244 B: They're asking what's the difference between the rectangle that was created and all this [points to the light grey area]

245 A: Ah, as long as the rectangle is small the difference between the areas is small

246 B: Close. We're saying: The change of the accumulation function with a smaller and smaller *h* is almost not different from the rectangle

The students combine (build-with) constructs including CAF (233) and RM (lines 230-231) and get (construct) the new one, iCAF. It is also interesting to mention the informal, intuitive and, in our eyes, appropriate use of the idea of limit (245-246). The students were then asked to decide how to approximate the rate of change of *Af* when *h* is very small. They tried algebraically using RCAF but were only able to simplify $x+h-x$ to h. They also tried visually, scribbling vertical dashes in the light grey area in Figure 1, but got stuck with this too. Only a strong hint of the researcher led to the student behavior we were looking for:

260 R: How do we calculate the area of this rectangle?

261 A: It's h times *f* of *x*

262 R: Does this help you with rate of change?

263 B: Like this? [writes *f*(*x*) times *h*]

After this, a rather long and somewhat confused episode with back-forward considerations followed, during which the students were trying to recognize the elements relevant to the situation, and to build with them what the task required, namely to approximate the rate of change of *Af* when *h* is very small. Our interpretation is that the origin of their difficulties is rooted in their limited RoC construct. Anyway, in the long run it seems that they recognized relevant constructs including iRCAF and DRoC and combined them to get a rather sophisticated conclusion:

320 A: … rectangle, *f x* times *h* because it's almost a change, partly *h*, right? What did we get?.. So that's *f* of *x*

324 B: The function. The derivative of the accumulation equals the function

336 A: That as long as you take a smaller h the area actually approaches to being the rectangle and this rectangle is actually equal to the function

$$\frac{A(x+h)-A(x)}{h} \approx \frac{h\cdot f(x)}{h} = f(x)$$

$$A'(x) = f(x)$$

While this does not fit the operational definition of the FTC construct yet, it does point to a beginning of its emergence.

DISCUSSION

In the study presented here, we tried to find an answer to the question how processes of knowledge construction occur during an instructional intervention in the form of a teaching interview on the FTC. We reported findings about one pair of students. We choose this pair because it represents the sample. The findings about other groups are quite similar. The differences (rate of progress, level of discussion, quality of previous knowledge, completeness of new constructs and so on) depend on the personal, cognitive and social context. The data gathered and their analysis using the RBC methodology allow us to give a tentative answer to the above question. From the excerpts we can see that, at the current stage of learning, the students succeeded to construct several elements of knowledge that were totally new to them. The analysis of the excerpts shows that the students recognize relevant previous constructs, build new constructs with them, and use these new constructs for progressing toward the FTC. The main conclusion we draw from this study is that the notion of accumulation has allowed us to design a didactical tool to support students' knowledge constructing processes on integration, and that the adopted research methodology has allowed us to observe these processes. An additional conclusion is that the understanding of the FTC using this approach depends on students' understanding of accumulation as well as on their understanding of rate of change. Despite the rather impressive general performance of the students, we feel that their construct of rate of change was inappropriate or too fragile. The RBC methodology allowed us to note this weak link in the chain. The next question we hope to research is whether, in the high school context, advanced concepts like accumulation and rate of change might be introduced separately, one after another, or, perhaps, combined in some dialectical didactical entity.

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