CHOOSING AND USING EXAMPLES: HOW EXAMPLE ACTIVITY CAN SUPPORT PROOF INSIGHT

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This paper presents the results of two studies aimed at identifying the ways in which successful provers (students and mathematicians) engage with examples when exploring and proving conjectures. We offer a framework detailing the participants' actions guiding a) their example choice and b) their example use as they attempt to prove conjectures. The framework describes three categories for example choice (choose examples that test boundaries, emphasize mathematical properties, and build a progression of example types) and three categories of uses (identify commonality, see generality, and anticipate change).

INTRODUCTION: PROOF IN SCHOOL MATHEMATICS

Proof in school mathematics plays an important role in students' mathematical reasoning abilities, with researchers arguing that proof should be a central part of students' education at all levels (e.g., Ball et al., 2002). Yet despite a strong emphasis on proof in school mathematics, students struggle both to produce and understand mathematically valid proofs (e.g., Harel & Sowder, 1998; Healy & Hoyles, 2000). Researchers suggest that a critical source underlying students' difficulties is their treatment of examples, particularly the tendency to rely on example-based arguments as justification that a universal statement is true (e.g., Healy & Hoyles, 2000; Knuth, Chopin, & Bieda, 2009).

Although it is important to help students understand the limitations of example-based arguments, we propose that it is equally important to avoid situating example-based reasoning solely as an obstacle to overcome. Given the essential role examples can play in exploring conjectures and developing proofs, we suggest that providing students with opportunities to carefully analyse examples may contribute to their abilities to develop and understand conjectures and proofs. This paper presents the results of two studies aimed at identifying the ways in which successful provers (students and mathematicians) engage with examples when exploring, understanding, and proving conjectures. We offer an initial framework detailing the characteristics of participants' example choices and their example usage as they explore conjectures and develop deductive proofs. By studying the thinking of those who are successful at proving, our aim is to gain insight into the nature of the type of example-related activity that could ultimately support students' proof development.

BACKGROUND AND THEORETICAL FRAMEWORK

The Role of Examples in Conjecturing, Generalizing, and Proving

Examples play an important role in mathematical reasoning, and the time spent analysing examples can provide both a better understanding of a conjecture and insight into the development of its proof (Epstein & Levy, 1995). Thinking with examples can help students make sense of conjectures and can support the development of conceptual understanding more generally (e.g., Alcock & Inglis, 2008). Example use has also been found to support students' acts of generalizing (e.g., Goldenberg & Mason, 2008; Naftaliev & Yerushalmy, 2011), and analyzing structural similarities and variation across examples can support proof development (Goldenberg & Mason, 2008; Pedemonte & Buchbinder, 2011).

Research on mathematicians' thinking similarly shows that examples play a critical role in both mathematicians' development of conjectures and in their subsequent construction of proofs (Alcock & Inglis, 2008). Epstein and Levy (1995) contend that mathematicians spend considerable time thinking with examples, noting, "It is probably the case that most significant advances in mathematics have arisen from experimentation with examples." (p. 6) This current study builds on prior work (Lockwood et al., 2012) in which mathematicians described using examples specifically to gain insight into proof. While the initial research on example use shows promise, more nuance is needed in understanding how to best support students' thinking with examples in order to promote proof. The findings presented in this paper shed light on the specific mechanisms through which example exploration provides insight into proof development for both students and mathematicians.

Proof and Proof Activities

We refer to proof and justification interchangeably to mean the activity of ascertaining (convincing oneself) and persuading (convincing others) (Harel & Sowder, 1998). An individual's proof scheme consists of what constitutes ascertaining and persuading for that person. We rely on Harel and Sowder's (1998) proof schemes framework – recently updated (Harel, 2007) – for classifying students' proof schemes. The framework establishes three main classes of proof schemes: (a) External Conviction class, (b) Empirical class, and (c) Deductive class. Proof schemes in the first two classes rely on external authority, the appearance of an argument, manipulation of symbols without a coherent system of referents, or evidence from examples. In contrast, the deductive class of proof schemes represents schemes dependent on generality, operational thought, and logical inference.

METHODS

Student Study: Participants and Instrument

Participants were 20 students aged 12-14, each who participated in a videotaped 1-hour interview. Eleven students were female and 9 students were male. The interview instrument presented students with seven conjectures and students were asked to examine the conjectures, develop examples to test them, and then provide a justification. The conjectures addressed ideas in number theory and geometry. A sample conjecture is as follows: "Kathryn thinks this property is true for every whole number. First, pick any whole number. Second, multiply this number by 2. Your answer will always be divisible by 4." After the students worked with examples for each of the conjectures, they were asked why they chose the examples they did.

Mathematician Study: Participants and Instrument

Participants were 6 male faculty members from two university mathematics departments who participated in 1-hour videotaped interviews. Five participants hold PhDs in mathematics and one in mathematics education. During the interviews, mathematicians were asked to explore three mathematical conjectures and to think aloud as they worked. After each conjecture, the participants were asked clarifying questions about their work, including their example-related activity. Sample conjectures are shown in Table 1.

1	Let S be a finite set of integers, each greater than 1. Suppose that for each integer n there is some $s \in S$ such that $\gcd(s,n)=1$ or $\gcd(s,n)=s$. Show that there exist $s,t \in S$ such that $\gcd(s,t)$ is prime.
2	All the numbers should be assumed to be positive integers. An abundant number is an integer n whose divisors add up to more than $2n$. A perfect number is an integer n whose divisors add up to exactly $2n$. A deficient number is an integer n whose divisors add up to less than $2n$. Conjecture 2a. A number is abundant if and only if it is a multiple of 6. Conjecture 2b. If n is deficient, then every divisor of n is deficient.

Table 1: Sample conjectures given to mathematicians

Analysis

The justifications that the student participants produced were coded according to Harel and Sowder's (1998; Harel, 2007) proof schemes taxonomy and identified as representing proofs from the deductive class, the empirical class, or the external conviction class. Five of the 20 student participants produced no proofs from the deductive class, and 11 participants produced some proofs from the deductive class. Because the paper focuses on successful provers, the student analysis was restricted to the remaining 4 students who produced proofs that were all from the deductive class, and all of the mathematician interviews were analysed. Each of these 4 students was able to produce a deductive proof for every conjecture he or she encountered.

Both the mathematicians' and the students' examples were coded into a pre-existing framework of example *types* and *uses* developed by the authors (Ellis et al., 2012; Lockwood et al., 2012). We then re-analysed the data by coding the participants' responses to each conjecture in order to characterize common themes across their actions. This open coding process led to the development of two major categories of themes, example choice and example use. The research group discussed the codes and clarified uncertainties as emergent codes solidified. A given response could be coded in multiple categories simultaneously, both within and across choices or uses.

RESULTS: EXAMPLE CHOICE AND EXAMPLE USE

We identified two major actions with examples that supported the participants' proof activities: Deliberate and strategic *choice* of examples, and insightful *use* of examples. The categories of example choice and example use are shown in Table 2. Although every category occurred in both the mathematician data and the student data, due to space constraints we limit our discussion to the most salient examples.

Example Choice	*Test boundaries*: Selecting examples that target the boundaries of the hypothesis or conjecture, including counterexamples.
	Emphasize properties: Purposefully choosing examples with particular properties or features relevant to the conjecture in question.
	Build a progression: Building a deliberate progression of specific examples that may range in type or role.
Example Use	*Identify commonality*: Attending to common features or characteristics across multiple examples in order to identify a broader mathematical structure.
	See generality: Identifying a general or representative structure embedded in one example that may provide insight into the structure of a general argument.
	Anticipate and imagine change: Envisioning an example as a dynamic, changing representation.

Table 2: Categories of example choice and example use

Example Choice

The participants demonstrated a dispositional orientation towards choosing examples in a deliberate, strategic manner. The first category of example choice is *testing boundaries*, in which one purposely attempts to find examples that could potentially break the conjecture, or could provide insight into the conjecture's limitations. For instance, in the mathematician's Conjecture 2b, Professor Lowry specifically chose to explore examples that included 6 as a factor because 6 is a perfect number: "We know 6 is perfect…so actually it's a good choice for a potential counterexample, because it's not deficient, but it's not far from being deficient." He further clarified his motivation for this choice by saying it is "likely that if something interesting is going to happen with an example, a boundary case is usually where it would be interesting." Professor Lowry indicated that by examining boundary cases and looking for counterexamples, he suspected that he might gain some insight about the conjecture and about a possible proof. The student participants also selected examples with boundary testing in mind. For instance, Genna examined Conjecture A, the conjecture that the sum of the lengths of any two sides of a triangle is greater than the length of the third side. Genna believed that this must be true for equilateral and isosceles triangles because she could imagine that "two of the sides added together are obviously bigger than the third side." Genna then tried to think of "a triangle that wouldn't work", and drew a scalene triangle. Genna's explorations with different scalene triangles led to a general argument for all triangles. In general, successful provers recognized what they could gain from boundary cases and specifically sought out those types of examples.

Participants also chose examples based on their mathematical *properties*, the second category of example choice. When doing so, participants sought out examples with specific types of properties that they viewed as relevant to the conjecture at hand. This was evident in Genna's selection of a scalene triangle, as she thought that triangles with three different side lengths might be more likely to break the conjecture than equilateral or isosceles triangles. An attention to properties emerged frequently for the mathematicians. In his work on Conjecture 1, Professor Parker drew upon the specific mathematical property of relative primeness in ascertaining the truth of the conjecture: "See I was definitely using relatively prime to this [circles a 4], relatively prime to this [circles a 6], giving me the existence or non-existence of a two and a three." In his work, mathematical properties such as prime numbers, greatest common divisors, and relative primeness were readily available to him, and he referred to them often in selecting examples.

The final category of example choice is *progression*, which describes an attempt to build a set of examples that either varies across different types or roles, or that together contributes to a more complete picture of what is happening. For instance, for Conjecture 2 Professor Lowry chose examples with 6 as a factor. He first specifically chose the example 12, which is 6•2, but he recognized the fact that because 6 was half of 12 he would never attain a counterexample. He thus proceeded to choose an example that did not have a perfect factor that was exactly half of it, selecting 6•3. As he worked through this example, though, he realized that many other factors were being generated. He did not want to have "much stuff between six and the whole number," and this led him to choose 6•11. In the end, it was his work with the 6•11 example that led him to a proof of the conjecture. Professor Lowry's strategic and carefully chosen progression of 6•2, 6•3, and 6•11 provided him with a number of insights and ultimately contributed to his successfully proving the conjecture.

Example Use

The participants' dispositional orientation towards using examples reflected the belief that the purpose of example exploration is not merely to check a conjecture's truth, but to try to understand a conjecture's logic through the example. In the first category of use, *identify commonality*, participants paid careful attention to the variation across multiple examples, attending to what changed and what remained the same as they shifted from one example to the next. For instance, Professor Larkin noted that he wanted to be aware of patterns emerging in multiple examples, in particular "what pattern it's creating for me. So that, if in fact, the [conjecture] is true, I have some sense of the pattern I can create to prove it." In his work on Conjecture 1, Professor Willis also mentioned seeking a pattern across examples, suggesting that he was looking to identify a common structure among his chosen examples that would shed light on why the conjecture might be true.

In the second category of example use, *see generality*, participants were able to develop insight into the mathematical structure of a potential argument through exploration with just one example. This way of thinking is evident in Reed's exploration of Conjecture B, the conjecture that the sum of three consecutive whole numbers is equal to three times the middle number. Reed tested with the triple 33, 34, 35. When the interviewer asked if that example worked, he replied, “Yes it did. That’ll do. Well, of course it’ll always work.” Reed underlined the middle number, 34, and appeared to experience an insight into why the conjecture must always work. When the interviewer asked him to explain, Reed used a new triple, 6, 7, and 8, to explain his insight (Figure 1):

Figure 1: Reed’s example to demonstrate the truth of Conjecture 1

Reed: Eight minus 1 equals 7, and 6 plus 1 equals 7. So take 1 off this (gestures to the 8) and put it on there (gestures to the 6). And it comes out 7 plus 7 plus 7.

The final category of use, *anticipate and imagine change*, refers to an ability to imagine one example as dynamic rather than static, changing the boundaries or features of the example in order to mentally anticipate and test multiple cases at once. In some cases this can also enable participants to deliberately manipulate the example in a way that can assist with insight into a proof. This is seen in Reed’s work with Conjecture A. Reed constructed several triangles to try to imagine whether it would be possible to have two sides longer than the third, which he began to suspect might be impossible. In order to confirm his suspicions, Reed created a final triangle with two sides each of length 4 and the third side of length 8, explaining what would happen if he “straightened” the two sides of length 4, by which he means flattening the sides so the triangle became more and more obtuse until it approximated a line:

Reed: Because if these sides were straightened out to make a line, it’d be this long (gestures a length from A to C, a length longer than the 3rd side) so this line right here from point A to point B is not the same as – the same or longer than points A to C.

The participants’ orientation towards example use was one that cast examples as a way to better understand the conjecture. In contrast, students who were unable to develop deductive proofs viewed examples primarily as a way to test a conjecture’s truth. By moving beyond testing activities, the successful provers in this study were able to leverage the power of examples to provide meaningful insight into the conjectures and their potential proofs.

DISCUSSION

Although the students and the mathematicians differed in the sophistication of the arguments they were able to construct, there were a striking number of similarities in the ways in which each participant group thought strategically about their example choice and made deliberate use of examples in order to think about broader mathematical structures. Their ways of choosing and using examples differed from what occurred in the work of the student participants who did not ultimately produce deductive arguments (see Ellis et al., 2012). These differences suggest that exposure to examples is not sufficient for fostering proof insights; instead, learners must engage with examples in particular ways in order to benefit from their utility as a way to gain understanding and inform the development of deductive arguments.

In addition, there were some important differences across the two participant groups; for instance, the mathematicians were more apt to recognize the potential power of a specific example before choosing it, and they demonstrated an explicit meta-cognitive awareness of the usefulness of examples more generally in providing insight into the nature of a conjecture and its proof. These findings, rather than framing examples as obstacles to overcome, emphasize that students may benefit from instruction in how to strategically choose examples and how to think carefully with the examples they have chosen. Further, instructional practices that encourage students to discuss and justify their choice and use of examples could foster the development of the meta-cognitive awareness demonstrated by mathematicians. A stronger understanding of the strategies successful provers employ as they use examples to create, explore, and prove conjectures could ultimately inform instructional guidelines aimed at more effectively fostering students' abilities to prove.

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