RETHINKING AND RESEARCHING TASK DESIGN IN PATTERN GENERALISATION

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This paper describes the design principles behind a test instrument, called the JuStraGen test, that had been specially developed to assess students' ability to generalise figural pattern generalising tasks, as well as to measure the effects of two task features on their rule construction. A discussion of some student responses then follows to shed light on how students dealt with some tasks in the test.

BACKGROUND

Several past studies on pattern generalisation have reported low success rates for figural generalising tasks in which just a single configuration was presented. In a study by Hoyles and Küchemann (2001), nearly 2800 high attaining Year 8 students in the UK were asked in one of the tasks to inspect a single generic case in order to find the number of grey tiles needed to surround a row of 60 white tiles. The border-tiling task had a success rate of 42%, which was considerably low taking into account the students' prior attainment. The same border-tiling task was also used by Cañadas, Castro and Castro (2011) on over 350 Years 9 and 10 students in Spain, this time asking them for the number of grey tiles needed to surround a row of 1320 white tiles. The success rate of about 55% was similarly moderate. Like these researchers, Steele (2008) had rather limited success in getting students to work out a functional rule for predicting the number of blocks in a staircase with *n* steps in a classic *Staircase* task that shows only a four-step-high staircase. Six of the eight students in the US had difficulties constructing the functional rule, which was quadratic. The type of function in this task might well have been a contributing factor. In another "classic" matchstick task that appeared in TIMSS–2007, a single configuration showing a row of four squares made of 13 matchsticks was provided and Year 8 students were asked about the number of squares in a row that could be made using 73 matchsticks. The success rate for Year 8 students internationally was barely 9% compared to about 41% for Singapore students (Foy & Olson, 2009).

The success rates of students in the abovementioned studies clearly indicate that the rule construction process in pattern generalisation is often fraught with difficulties, with many students often failing to navigate this process successfully. Such difficulties could be attributable to several student-related factors, ranging from ignorance of appropriate generalising strategies (Moss & Beatty, 2006) to lack of spatial visualisation techniques (Warren, 2005) and inexperience in using the highly specific mathematical language of algebra to express generality (Hoyles, Noss, Geraniou, & Mavrikis, 2009). But in the light of the earlier paragraph, we posit that student difficulties in rule construction might also be triggered by task features, in particular,

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two features that we categorise as *the format of pattern display* (Chua & Hoyles, 2012) and *the type of function*. Revisiting the *Staircase* task in Steele's (2008) study for example, we wonder if the students' difficulties were influenced by the provision in the task of only a single generic case, or by the quadratic nature of the underlying pattern. Whether Steele realised these potential obstacles in her task is unclear, but Küchemann (2010), however, firmly maintains that the factor contributing to student difficulties in that task was the format of pattern display, and not the type of function.

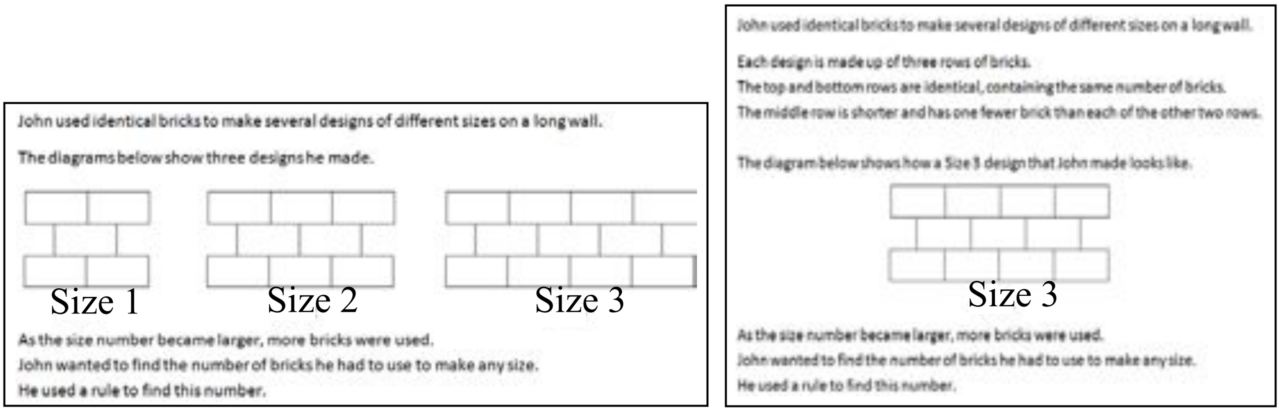
Our present study, therefore, aims to examine systematically the effects of different formats of pattern display and types of function on students' pattern recognition and their ability to generalise. In order to carry out the study, it was first necessary to construct and validate an instrument; in this case a specially-designed paper-and-pencil test. In this paper, we address the following question: What task design considerations were taken into account when devising the test instrument? In what follows, we describe the development of the test instrument, present some test items and highlight some student responses to illustrate its implementation.

DEVELOPMENT OF TEST INSTRUMENT

We were unable to identify from the review of the research literature a test instrument that would characterise the effects of task features on students' pattern recognition and their ability to generalise. We therefore set out to design a new test instrument, which we entitled *Strategies and Justifications in Mathematical Generalisation* (*JuStraGen*). It was developed specifically to achieve the aims of the present study.

The *JuStraGen* test was designed to provide an assessment of students' ability to generalise figural pattern tasks, as well as a measurement of the effects of two task features on their rule construction. It is a paper-and-pencil test consisting of eight generalising tasks designed to investigate how students construct and justify the functional rule for predicting any term of a pattern in the tasks. Of the eight tasks, the underlying pattern structure was linear for four of them and quadratic for the other four. Furthermore, the test was also developed specially to examine systematically the effects of *the format of pattern display* (i.e., successive vs non-successive configurations) and *the type of function* (i.e., linear vs quadratic) on students' ability to construct the functional rule. Figure 1 shows a linear task in the two different formats. Students were required individually to work out a functional rule for the pattern in terms of the size number, and justify how they obtained the rule.

To examine whether different formats of pattern display had any effect on students' rule construction, we chose to use a between-subjects experimental design involving two groups, Group 1 (G1) and Group 2 (G2), of students. As for testing whether different types of function had any effect on students' rule construction, a within-subjects experimental design was adopted. In short, G1 worked on both linear and quadratic generalising tasks with successive configurations whereas G2 was given identical tasks but with non-successive configurations.



(a) Successive format (b) Non-successive format

Figure 1: Bricks

The development of the *JuStraGen* test was guided by the research design above and the following general considerations.

(a) ***Number of generalising tasks*** Deciding on how many tasks to set is a tricky matter: too few tasks may limit the generalisability of the results about the effect of task features on students' success in establishing the functional rule; whilst having too many tasks is simply not practical given the time needed to complete them. After pre-piloting a task to gauge the amount of time students needed to complete it, we decided to set eight tasks. We believed that this number of tasks was a reasonable figure for covering a range of non-successive configurations.

(b) ***Task scenario*** Most figural generalising tasks used in research and in textbooks ask students to consider a sequence of configurations and then make some near and far generalisations, followed by finding the rule underpinning the pattern depicted in the sequence (see Rivera & Becker, 2008). The tasks rarely provide a scenario in which the purpose of representing the pattern with a functional rule might be apparent. For some students, it might, therefore, be difficult to see *why* they have to do what is required of them. To provide some impetus for students, we tried to adopt the notion of *purpose* (Ainley, Pratt, & Hansen, 2006) to make the tasks as meaningful as possible for the students. We framed the generalising tasks in different scenarios, such as making wall designs for *Bricks*, and stated the motive as wanting the students to help the character in the task to find the rule for constructing any size (e.g., John wanted to find the number of bricks he had to use to make any size in *Bricks*. Write down the rule John might have used in terms of the size number).

(c) ***Parallel tasks*** To determine whether the format of pattern display influenced the students' construction of the functional rule, each task was created in two different formats, with its pattern depicted as (1) a sequence of three successive configurations, and (2) a single configuration or a sequence of two or three non-successive configurations. For instance, the *Bricks* task in Figure 1 above shows three configurations (Sizes 1, 2 and 3) for the successive format and a single generic configuration (Size 3) for the non-successive format.

(d) ***Matching tasks*** To determine whether the type of function influenced the students' construction of the functional rule, each linear generalising task had a matching quadratic generalising task. Table 1 below lists the matching linear and quadratic generalising tasks, with details about the format of pattern display. For each pair of tasks, the description of the scenario was kept invariant: for instance, both *Bricks* and *Wall Design* were set in the same scenario of creating wall designs using bricks. Furthermore, the shape of the configuration in each linear task was created to resemble as closely as possible that of the matching quadratic task. Considering the *Birthday Party Decorations* and *Christmas Party Decorations* tasks for example, both sets of configurations look alike except for the blocks in the middle. We believe that careful considerations to such details during the task design process are essential as pre-emptive measures for minimising the possible interference of task scenario on the outcome of the *JuStraGen* test so that more robust conclusions can be drawn about the effect of the type of function on how students construct the functional rule.

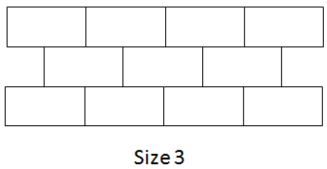
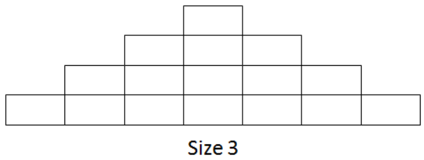
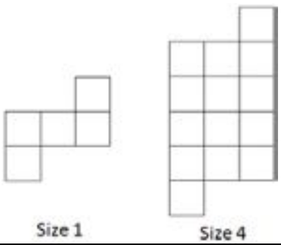
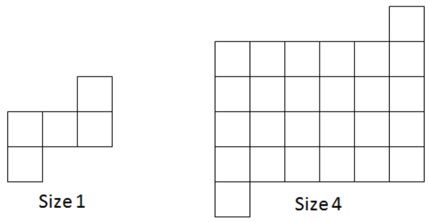
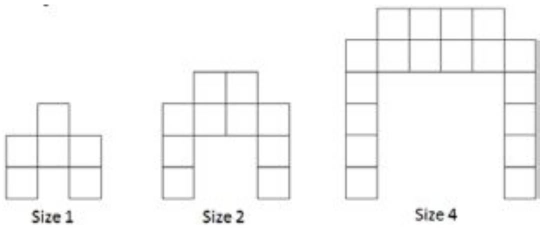
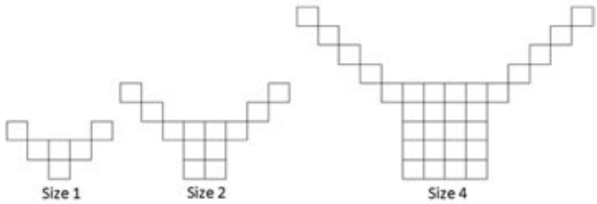
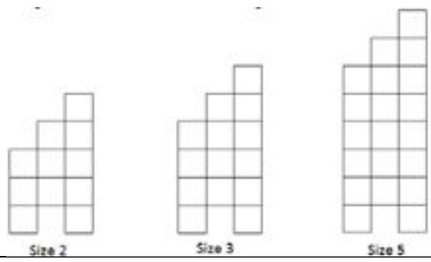
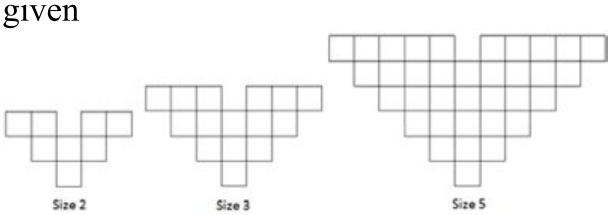
Linear	Quadratic
Bricks For successive format: Sizes 1, 2, 3 were given 	Wall Design For successive format: Sizes 1, 2, 3 were given 
Birthday Party Decorations For successive format: Sizes 1, 2, 3 were given 	Christmas Party Decorations For successive format: Sizes 1, 2, 3 were given 
Towers For successive format: Sizes 2, 3, 4 were given 	Oh Deer! For successive format: Sizes 2, 3, 4 were given 
High Chairs For successive format: Sizes 2, 3, 4 were given 	Tulips For successive format: Sizes 2, 3, 4 were given 

Table 1: Matching linear and quadratic generalising tasks

Bricks, *Birthday Party Decorations*, *Towers, High Chairs*, *Oh Deer* and *Tulips* were six new generalising tasks designed specially for the *JuStraGen* test. *Christmas Party Decorations* and *Wall Design* were adapted from studies by Rivera (2007), as well as Smith, Hillen and Catania (2007).

(e) ***Number of non-successive configurations*** In order for students to move to articulating the functional rule underpinning a pattern, we notice from the literature review that there are two common approaches in figural generalising tasks: first, to provide three configurations (see Rivera & Becker, 2008; Smith et al., 2007); and second, to show just a single configuration to represent a generic case of the figural pattern, as we have discussed previously. What is less common in the literature, however, is the use of two configurations. So far, we have found only three studies using it. The *Ladder* problem in Stacey's (1989) study showed two successive configurations whereas Healy and Hoyles (1995), as well as Warren and Cooper (2008), used two non-successive configurations in their studies. All these studies provided little, if any, explanation of the rationale for choosing to use these numbers of configurations. But, nonetheless, these numbers do appear to be sufficient to allow students to detect the pattern and then construct the rule. So we can infer that having more configurations would not make any difference. Guided by the outcome of the literature review, the present study decided to use one, two or three non-successive configurations in the *JuStraGen* test.

One might now ask whether it is really possible to discern the underlying pattern structure from just a single configuration. To address this concern, it was important to offer a *general* description of the single configuration. Although the description provided essential information for students to realise how the pattern would grow, it did not disclose the functional rule underpinning the pattern however. Furthermore, the use of a single configuration was limited to only one pair of generalising tasks – *Bricks* and *Wall Design*.

No description of the configuration was given for the remaining pairs of generalising tasks. Like single configuration, the use of two non-successive configurations was also limited to one pair of tasks – *Birthday Party Decorations* and *Christmas Party Decorations*. Three configurations were provided in *Towers* and *Oh Deer*, as well as in *High Chairs* and *Tulips*.

(f) ***Structure of task*** All the generalising tasks were unstructured in order to allow students scope for exploration so that they could come up with their own interpretations. This would allow us to see how the students came to recognise and perceive the pattern without scaffolding. so there were no part questions asking for near or far generalisations that would gradually lead students to detect and construct the functional rule underpinning the pattern.

(g) ***Size number of configurations*** The size numbers of the three given successive configurations ran from either Size1 to Size 3 or Size 2 to Size 4. As for the non-successive format, any single configuration starting from Size 3 was thought to be a reasonable generic case for representing a pattern. Thus Size 3 was given in *Bricks* and *Wall Design*. Warren and Cooper (2008) used solely odd-numbered sizes (Sizes 1 and 3, and Sizes 1, 3 and 5) in two of their tasks that involved two or three non-successive configurations. Their choice of configurations, we would argue, might be unfortunate because students might think that the even-numbered sizes did not exist in these tasks. So for generalising tasks with two or three non-successive configurations, we believed it was important to include both odd-numbered and even-numbered sizes so as not to mislead any students into thinking that certain sizes did not exist in the pattern. Therefore, we included both Sizes 1 and 4 in *Birthday Party Decorations* and *Christmas Party Decorations*. In a similar vein, Sizes 1, 2 and 4 were shown in *Towers* and *Oh Deer*, and Sizes 2, 3 and 5 in *High Chairs* and *Tulips*.

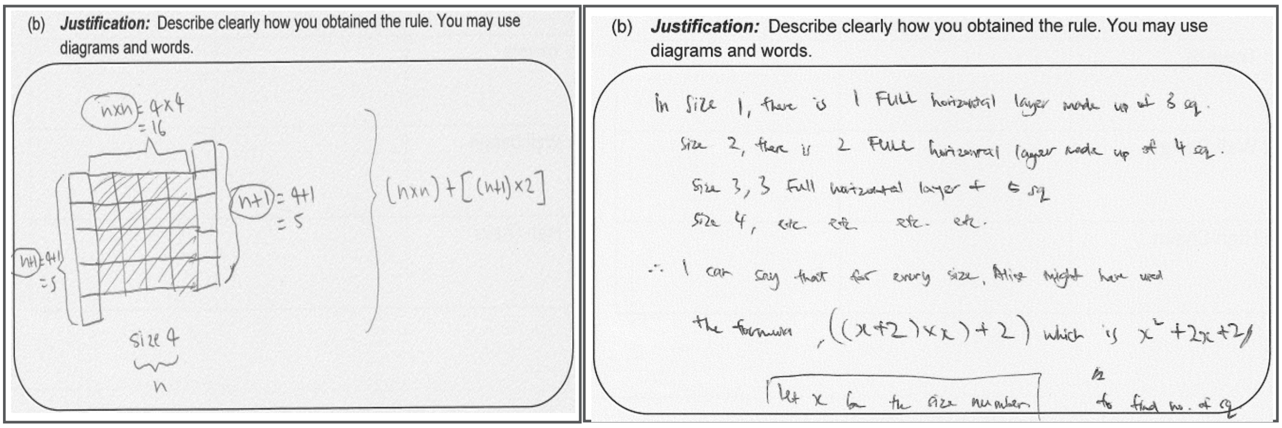
(h) ***Shape of building materials*** Square cards or tiles and rectangular bricks were used to build the configurations. Other shapes such as circles and triangles were omitted in order to eliminate the confounding influence of the shape used to build the configurations on students' ability to generalise.

This section highlighted the key design principles that we had applied to develop carefully crafted generalising tasks for the *JuStraGen* test. Due to limited space here, we will briefly describe what some students actually did when dealing with the non-successive tasks in the next section.

SOME FINDINGS AND CONCLUSION

56 G2 students (aged 14 years) from a secondary school in Singapore were administered the *JuStraGen* test with non-successive configurations. Having learnt the topic of number patterns before sitting the test, the students should be able to continue for a few more terms any pattern presented as a sequence of numbers or configurations, make a near and far generalisation and establish the functional rule in the form of an algebraic expression.

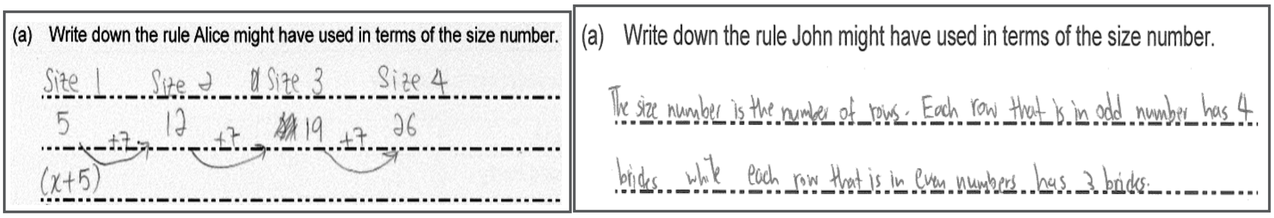
The unstructured nature of the generalising tasks allowed students plenty of scope for developing their answers. When a single or two configurations were given, some students had to work out other configurations before they could see the structural relationship from the geometrical arrangement of tiles or cards (see Figure 2(a)). For some other students, finding additional configurations was not necessary at all as they were able to abstract the structural relationship from the given diagrams by treating them generically (see Figure 2(b)). Therefore, students' ability to derive the functional rule was clearly assisted by their *awareness* of the structure inherent in the pattern, and *not* the format of pattern display. We consider this finding very encouraging, knowing that our decision to design unstructured tasks was appropriate.



(a) with additional configurations (b) without additional configurations

Figure 2: Recognising structure in Christmas Party Decorations

Students' inability to recognise the pattern underpinning a single or two configurations is not totally unexpected and, in particular, two student responses are worth discussing with respect to our design principles. Figure 3(a) shows a student misinterpreting the *Bricks* pattern despite the provision of a description of the configuration. For this student, the number of rows in a configuration corresponds to its size number, and the number of bricks per row alternates between four in odd rows and three in even rows. Although there were only six such cases (11%) in the present study, the frequency of cases could have been higher if the task had not provided the description. We are, therefore, now convinced that the inclusion of a general description of the single configuration in the *Bricks* task was crucial and necessary.



(a) Bricks (b) Christmas Party Decorations

Figure 3: Student errors

The response in Figure 3(b) shows a wrong pattern produced by a student for the *Christmas Party Decorations* task. Somehow the student must have inspected the difference between the given Sizes 1 and 4, then figured out that the difference could be evenly divided over four successive configurations. This discovery eventually led to working out Sizes 2 and 3. The numerical terms {5, 12, 19, 26} did not match the figural pattern even though they formed a linear sequence. Its validity could have been easily verified by drawing out the configurations for Sizes 2 and 3. We want to argue that the student's error is *not* caused by any design flaw in the task but by the student himself or herself for making a wrong assumption about the pattern and using an inappropriate strategy (i.e., finding the common difference).

To conclude, this paper introduced the *JuStraGen* test instrument that was developed from scratch to serve the purposes of the present study. We hope the detailed description of the design of the test instrument will permit other researchers to use the instrument in the same way as we used it and to further develop it so it can serve as a useful tool for the community.

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