CROSSING THE BORDERS BETWEEN MATHEMATICAL DOMAINS: A CONTRIBUTION TO FRAME THE CHOICE OF SUITABLE TASKS IN TEACHER EDUCATION

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In mathematics teaching, closed boundaries between mathematics domains may convey to students a "fossilized" image of mathematics, and, in turn, cause difficulties in problem solving. Teachers should promote a "crossing" perspective in their teaching. In order to make teachers able to cross the borders, teacher education must encompass suitable tasks to be experienced and discussed. This paper reports a study aimed at framing the choice of such tasks, and the analysis of related problem solving behaviors. A contribution to framing comes from an adaptation of Habermas' construct of rational behavior. An experimental situation is planned and analyzed according to the resulting framework. Some educational implications are derived.

INTRODUCTION

Usually, mathematics is taught in secondary school as a discipline divided into separate domains (algebra, geometry, analytic geometry, etc.), each of them with specific theories, problems to solve and tools to solve them. This situation conveys a "fossilized" image of school mathematics and is one of the causes of students' difficulties in complex pure and applied mathematical problem solving. The traditional organization of mathematics teaching at university level, with reference to curricula designed for prospective mathematics teachers, may reinforce such kind of teaching and even justify it at the teachers' eyes - against present trends in the development of pure and applied mathematics, where frequently problems in a domain are tackled by using tools and strategies belonging to other domains.

Two questions arise: how to promote student-teachers' awareness of the nature of mathematics (in particular, the potential inherent in crossing the boundaries between different domains, according to the needs of problem solving activities); and at the same time: how to prepare them to teach mathematics avoiding the usual closed compartments of teaching? According to a position shared by most mathematics educators (see Watson & Sullivan, 2008), prospective teachers should experience meaningful activities that allow them to develop awareness about crucial aspects of mathematics, and at the same time suggest them suitable tasks/methodologies for teaching, focused on those aspects. In our case a natural, initial step might be to allow student-teachers to experience, compare and discuss different ways (different tools and strategies, belonging to different domains of mathematics) of dealing with a task in pure or applied mathematics.

In this perspective, in order to plan and analyze suitable didactical situations we think that it would be useful to elaborate a framework including:

- a description of the specific competencies (concerning the specificities of the different areas of mathematics and possible links to establish between them) prospective teachers should develop;

- an analytical tool allowing to characterize and compare specific validation criteria of statements and inferences, specific problem solving strategies and specific ways of communicating in different domains of mathematics;

- an appropriate methodology to create and satisfy the need for detecting and comparing the specificities of the different domains of mathematics, in particular when involved in the solution of the same problem.

As concerns the first point, an initial contribution by the first two authors concerns the Cultural Analysis of the Content to be taught as a competence of epistemological, historical and anthropological analysis to be gradually acquired by teachers (see Boero & Guala, 2008).

As concerns the second point (related to the subject of this report), we can rely upon the previous work by the first and the third author who adapted Habermas' construct of rational behavior in discursive practices as a tool to plan and analyze some mathematical classroom activities: conjecturing and proving (see Boero, Douek, Morselli & Pedemonte, 2010); and the use of algebraic language in proving and mathematical modeling (see Morselli & Boero, 2011).

The aims of the study reported in this paper are: to further develop the adaptation of Habermas' construct of rational behavior, as a tool to characterize and compare the "rationalities" of different domains of mathematics; and to ascertain if the adapted construct can be exploited to plan and analyze suitable classroom activities aimed at preparing teachers to cross the closed borders of mathematical domains, thus contributing to the third point.

THEORETICAL FRAMEWORK

The interest of adapting Habermas' construct of rational behavior in discursive activities (as an analytical tool to characterize and compare the "rationality" of different domains of mathematics) depends on the fact that it consists of three inter-related components, which can be referred to three crucial and deeply interconnected aspects of mathematical activities:

- epistemic rationality (ER), consisting in the conscious validation of statements according to premises, true statements and inference rules that are shared in the reference community;

- teleological rationality (TR), consisting in the conscious choice of strategies and tools to achieve the aims of the activity;

- communicational rationality (CR), consisting in the conscious choice of suitable means to share ideas, problems, solutions in the reference community.

In Italy and in other countries two main subjects of secondary school mathematics are: synthetic (in particular, Euclidean) geometry; and analytic geometry. Using the adapted construct of rational behavior, we can distinguish between:

- a rational behaviour of synthetic type (briefly: synthetic rationality), according to the model of Euclidean geometry (but it can be transferred to other geometries too), based on strategies (TR) referring to visual evidence, aimed at proving the truth of statements and the validity of geometrical constructions (ER) through the construction of deductive chains based on axioms and previously proved statements; natural language not only plays a crucial communicational (CR) and reflective role, but also a treatment role to validate statements (ER: deductive chains mostly consist of verbal statements enchained through verbal links);

- a rational behavior of analytic type (analytic rationality), rooted in Greek mathematics (Menaecmus and Apollonius) and firstly made explicit in general by Descartes (1637; 1979), which consists (TR) in the construction of appropriate equations expressing the links between the relevant variables of the problem to be solved as if it would have been already solved, and in the solution of those equations - the resulting values of the unknowns allow to identify the solutions of the problem. We can further extend the scope of analytic rationality from the use of algebraic equations to include the use of calculus tools (like the use of the derivative to deal with tangent lines). In the case of analytic rationality natural language in its mathematical register works as a communicational (CR) and reflective tool, while algebraic language plays the major role in treatment, and epistemic control (ER) is exercised on the construction, transformations and interpretation of algebraic expressions (for details on ER in analytic rationality, see Morselli & Boero, 2011).

The aforementioned distinction guided us in planning a specific teacher education task; we have tested it in a selection process, thus in a situation far from an educational perspective (but, as we will see in the last section, the discussion of the task was an occasion to stimulate a reflective and learning process for some candidates). With reference to the chosen task, the aims of this research report are: to illustrate potential, specific features and limitations of synthetic and analytic methods, corresponding to specific aspects of synthetic rationality and analytic rationality; to describe and interpret people's behaviors according to the adapted Habermas' analytical tool; and to derive some implications as concerns teacher education (in the perspective of a more flexible teaching of main subjects of secondary school mathematics curricula).

While the adapted Habermas' construct plays a major role in the a-priori cultural analysis of the task and in the a-posteriori interpretation of behaviors, for the evaluation of the distance between people's behaviors and the requirements of epistemic and teleological rationalities we will exploit other theoretical tools:

- the construct of figural concept (Fischbein, 1993), which may account for the difficulties met in the mastery of figures without relating them to properties and definitions, or in the mastery of formal definitions and representations without the support of figural evidence;

- the construct of procept (Gray & Tall, 1994), which in particular accounts for the different ways of dealing with equations like $y=ax^2+bx+c$ as processes or as symbolic objects representing geometric entities - and the related difficulties.

METHOD

The task at issue was administered to 35 candidates to become mathematics teachers in secondary schools; most of them had a master degree in mathematics; the others had a master degree in Engineering or in Physics, with a strong curriculum in Mathematics; some candidates had also a Ph. D. in Mathematics or in Physics. The selection had to result in the choice of 15 candidates, who will enter one-year intensive professional preparation (including courses of mathematics education and stages in the schools) to become teachers.

Candidates had already passed a national test (35 candidates at the Genoa University had been successful in that test, out of 76) based on 60 multiple-choice questions. The further steps in the selection process (in each university) included a written test based on open problems, and an oral discussion with the local Commission "starting from the discussion of the written test" (in the case of the Genoa university).

The following task was prepared as one of the three tasks for the Genoa university written test (the other two tasks concerned calculus and probability issues; three hours was the whole available time):

> To characterize analytically the set ***P*** of (non degenerated) parabolas with symmetry axis parallel to the ordinate axis, and tangent to the straight line $y=x+1$ in the point (1,2).
>
> To establish for which points of the plane does it exist one and only one parabola belonging to the set ***P.***
>
> To find straight lines that are parallel to the ordinate axis and are not symmetry axes of parabolas belonging to the set ***P.***

The formulation of the task, as well as the a-priori analysis, was guided by the aforementioned theoretical framework concerning different rationalities in different mathematical domains. In our intention, the formulation of the questions would have encouraged the adoption of analytic methods without preventing candidates from using synthetic considerations with heuristic/teleological and control/epistemic functions, or even to get the solution for the third question. Moreover the formulation of the first and third questions would have encouraged the use of the language and methods of analytic geometry without preventing candidates from using calculus tools (an alternative choice more oriented towards calculus would have been to use calculus terminology: "quadratic functions", "graphs", "graph slope", and so on).

In the a-priori analysis of the task, we had imagined that:

- candidates could have answered the first question by intersecting the straight line $y=x+1$ with a generic parabola of equation $y=ax^2+bx+c$ passing through the point (1,2), and imposing that the intersection points collapse in that point; but also calculus notions could have been used by considering the quadratic function $f(x)= ax^2+bx+c$ with two conditions: $f(1)=2$, $f'(1)=2$;

- candidates could have answered the second question by choosing a generic parabola of the set ***P*** and imposing the condition of passing through a generic point (x_0, y_0), and then finding algebraically for which points the coefficients of the parabola are determined in an unique way; synthetic geometry might have provided them with the conjecture that lines x=1 and y=x+1 had to be excluded, and/or with a possibility of checking the correctness of their algebraic solutions. Note that synthetic geometry was not suitable to easily answer the second question, because proof (that, given a point S out of the excluded lines, it belongs to one and only one parabola of the set ***P***) requires the use of geometric properties of parabola, which are usually neglected in Italy in the teaching of conic sections;

- candidates could have answered the third question: either by analytic methods (once answered the first question, they could have written down the equation of the symmetry axis, or imposed that the first derivative is zero, given the equation of the parabola); or (once answered the second question) by synthetic geometric considerations bringing to the exclusion of the line x=1 due to the fact that the tangent line in the vertex (1,2) should be perpendicular to the symmetry axis, against the given condition of tangency in that point to the line y=x+1.

Possible limitations inherent in the didactical contract (concerning the legitimacy of the use of methods not alluded to in the text of the task, a serious problem in the case of a selective task) were at least partly overcome by the comment of one member of the commission, who under request of a candidate made explicit the fact that "different methods may be used to answer each of the three questions".

After the written test, the discussion with candidates (the further, final step of the local selection process) concerned this task for a representative sample of about one half of them (the other candidates discussed the other tasks of the test). The discussion of the work done by them in the written test was organized according to a semi-structured interview, around one or two of the following issues:

- difference between analytic and synthetic methods to deal with the questions (in terms of strategies and criteria of validation), in particular as concerns the exclusion of the lines x=1 and y=x+1 for the second question, and the line x=1 for the third one;

- heuristic and control potential of synthetic methods;

- relationships between the method of collapsing the points of intersection straight line/parabola in the tangency point, and the method which exploits the derivative of the quadratic function.

For the present research, data at disposal are: candidates' written solutions; and the records of the discussions of the candidates with the Commission.

SOME RESULTS

In order to give an idea of the preliminary analysis of the solutions, the following table summarizes some crucial features of the solutions of the first 11 candidates. AnGeo, Calc, SynthGeo are for the respective methods, with brackets indicating traces and/or trials, not the main adopted method; A added to AnGeo means purely algebraic calculations, with no substantial reference to geometric properties of parabolas, tangency, etc; PF and F mean, respectively, partial failure (when only one part of the answer is provided and is correct) or total failure. -- means: question not dealt with by the candidate.

Candidate	Degree	First question	Second question	Third question
1	Eng	AnGeo; (Calc)	--	AnGeo, A; F
2	Ph.D.Math	AnGeo	AnGeo	AnGeo
3	Math	AnGeo	AnGeo; PF	AnGeo; PF
4	Math	AnGeo; (Calc)	--	AnGeo
5	Math	AnGeo; (Calc)	AnGeo; PF	AnGeo
6	Math	AnGeo; (Calc)	--	AnGeo
7	Math	AnGeo	SynthGeo	AnGeo
8	Phys	AnGeo; (Calc)	AnGeo	AnGeo
9	Math	AnGeo, A	AnGeo	SynthGeo
10	Ph.D.Math	AnGeo; (Calc)	AnGeo	AnGeo
11	Ph.D.Math	AnGeo; (Calc)	AnGeo	AnGeo; (SynthGeo)

Some results emerging from further qualitative analyses of available data are:

a) the very limited use of the synthetic method; only 6 candidates out of 35 proposed some very short arguments of synthetic type; we observe that the formulation of the problem does not encourage it, thus using it requires consciousness of its potential and limitations. Moreover almost all those who engaged in synthetic geometry activities were unable to develop a rational behavior on the sides of TR (parabolas are only sketched, with no relation with the algebraic expressions representing them and very weak traces of some properties of parabolas of the set ***P***) and ER (drawings are very poor, with no comment, and sometimes do not include the parabolas under the line y=x+1, thus they cannot be used to check the validity of results derived through analytic methods). In terms of figural concepts (Fishbein, 1993), we may say that the figural aspect prevails on the conceptual one, with lack of epistemic control on the drawings and the related geometric figures;

b) the lack of functional connections between ER and TR; once engaged in the analytic geometry method, the algebraic calculation of the solution brings to a result which is neither checked by coming back to parabolas and their geometric properties, nor referred to the initial aim of calculations and (in the case of questions 2 and 3) to previous results;

c) for most students, CR works well only on the side of analytic geometry and of calculus; some students produce sequences of algebraic calculations with very few and not always appropriate words to present their solutions;

d) in some cases, the purely algebraic management of the analytic geometry method (11 candidates out of 35 performed only algebraic treatment) prevents students from discovering mistakes or lacks in their conclusions (ER); in terms of procepts the symbolic expression: $y= ax^2+bx+c$ is only a formal expression, with no reference to the process of generation of a line in the Cartesian plane and to its result.

e) many students, during the a posteriori discussions, had difficulty in connecting a typical feature of TR in the case of the analytic geometry approach to the first question (collapsing the intersection points parabola/line $y=x+1$ in (1,2)) with the limit process encapsulated in the expression $f'(1)$. During the discussions, the fact that the secant line does not pivot around the point (1,2) seems to prevent candidates from seeing that the analytic geometry process results in the approach of the secant to the tangent line in the point (1,2), and thus in an alternative way to access the derivative $f'(1)$;

F) during the discussion, the authors noticed positive learning reactions by the candidates (in spite of the psychological stress, due to the selective character of the discussion); most of them were able to realize (even with evident surprise!), under the commission members' guide, that:

- the method of collapsing the intersection points between the line $y=x+1$ and the parabola into the point (1,2) is another way of generating the derivative of the quadratic function for $x=1$;

- synthetic geometry can work as a tool for conjecturing and for checking results of analytic methods for questions 2) and 3);

- synthetic geometry can also allow to answer question 3), once question 2) has been solved.

CONCLUSIONS AND EDUCATIONAL IMPLICATIONS

The first data analysis shows the potential of the adapted Habermas' construct to produce suitable tasks for putting into question the rigidity of the separation between different mathematical domains, and to analyze people's behaviors in terms of their distance from rational behavior. The rigidity inherited from secondary and university teaching of mathematics is revealed, in terms of ER, TR and CR components, through the difficulties of moving forwards and backwards between synthetic and analytic geometry considerations, but also of identifying relationships between processes in analytic geometry and in calculus. The only language used by most candidates at an enough satisfactory and precise level is the language of analytic geometry. Thus, the task might be a starting point for an activity of teacher education aimed at putting into evidence the negative consequences of the rigid separation (in mathematics teaching) between different mathematical domains, and the opportunities offered by crossing the boundaries between them through suitable tasks. A further development concerns the conception of a teacher education experience, starting from the experimented task. The task, indeed, might be suitable (as revealed during the discussion) to start a program of Cultural Analysis of the Content to be taught (on the epistemological side), in particular in terms of critical consideration of nature, potential and limitations of analytic and synthetic methods, and features of the related rationalities. With reference to this possibility, an open problem concerns the opportunity that the adapted Habermas' construct , introduced as a tool for the researcher, becomes also a tool for teachers to identify and compare specific features of activities in synthetic geometry and in analytic geometry (and in other domains too), and of synthetic and analytic methods to deal with problems like the one considered in this report.

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