THE CONJECTURING PROCESS AND THE EMERGENCE OF THE CONJECTURE TO PROVE

Mara V. Martinez, Wenjuan Li

University of Illinois at Chicago

Building on Boero's (2007) thesis that conjecturing and proving are inter-related crucial mechanisms of generation of mathematical knowledge, this report focuses on the conjecturing process. First, a map of students' conjecturing process was produced taking into account different elements (e.g., conjectures, etc.). Second, based on our analysis a definition of conjecturing process is proposed including what is entailed in this process (e.g., rejection of conjectures). Third, three types of conjectures were identified from a cognitive perspective: "competing conjectures", "dismissed conjectures", and "conjecture to prove". In fact, the analysis shows the emergence of the "conjecture to prove" as the resulting knowledge from the conjecturing process. Data was collected as part of a teaching experiment where students had the opportunity to learn about algebraic proof.

INTRODUCTION

In the U.S. proof is included almost exclusively in the geometry high school curriculum. Students are typically taught to prove using the two-column format where they have to write derived statements on the left column, and, "reasons" on the right column (for a detailed analysis see Herbst, 2002). Stemming from this, there are significant consequences in terms of students' learning given the kind of proving experiences they have. In fact, students are often deprived of learning about proof in domains other than geometry. Besides, their experience of "proving" is reduced to a format (i.e., two column format) that, at best, portrays proving as a linear process.

In a regular geometry class, theorems are given to students in their final and formal presentation. As if, the process by which it was produced hasn't happened at all. Statements are presented as decontextualized and unproblematic objects, as if the production of the statements themselves were not at the heart of the production of mathematical knowledge. In fact, it appears that the production of a conjecture and the construction of its proof were unrelated. Although, students are not given the opportunity to construct their own conjectures, they are required to produce a proof.

Taken together, the sole focus on geometrical proof, the two-column format, and the absence of conjecturing, it is imminent that we provide students opportunities to experience a more rounded conception of the proving process and its role in the production of mathematical knowledge. It is in this context that the teaching experiment was conducted: to provide students the opportunity to produce their own conjectures and learn about algebraic proof.

2010. In Pinto, M. F. & Kawasaki, T.F. (Eds.). *Proceedings of the 34th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, pp. 265-272. Belo Horizonte, Brazil: PME.

Consequently, this paper aims to systematically study the conjecturing process as a central stage in the production of mathematical knowledge from a cognitive perspective. First, a map of students' conjecturing process was produced taking into account different elements (e.g., conjectures, order of appearance, use of numeric examples, etc.). Second, based on our analysis a definition of conjecturing process is proposed including what is entailed in this process (e.g., rejection of conjectures and production of mathematical relations). Third, three different types of conjectures were identified from a cognitive perspective: "competing conjectures", "dismissed conjectures", and "conjecture to prove". In fact, the analysis shows the emergence of the "conjecture to prove" as the resulting knowledge from the conjecturing process. By providing a detailed description of the cognitive value of the conjecturing process, we will argue about its importance in the production of mathematical knowledge and its consequent incorporation in the mathematics curriculum.

FRAMEWORK

Proof

In our work we develop and study problem situations that can foster students' learning of algebraic proof. Our framework has been conceptualized by bringing together the work of Balacheff (1988); Hanna (1990); Arsac and his colleagues (1992); and Boero (2007). From the work of Balacheff (1988), we build on the idea that *proof* is an explanation that is accepted by a community at any given time. In Balacheff's work, this definition of proof is introduced together with that of *explanation* and distinguished from the more specific *mathematical proof*. An *explanation* is the discourse of an individual who aims to establish for somebody else the validity of a statement. The validity of the statement is initially related to the speaker who articulates it. A *mathematical proof* is a proof accepted by mathematicians—and thus, distinct from proof. As a discourse, *mathematical proofs* currently have a specific structure and follow well-defined rules that have been formalized by logicians. In our teaching experiment, we centered our work on the idea of *proof* –as opposed to mathematical proof—in Balacheff's sense. During our teaching experiment, the use of a finite set of examples to show that a universally quantified (for all) statement is true would not fit within the *proof* category in the classroom culture. However, the use of examples is central to the production, refutation, and verification of conjectures. Therefore, within the teaching experiment the teacher negotiated with students what constituted a *proof* in that classroom, and what could be done by using a finite set of examples (e.g., production of conjectures).

We also drew from Hanna's (1990) distinction between *proofs that (just) prove* and *proofs that (also) explain*. The first kind *just* establishes the validity of a mathematical statement. The second kind, in addition to proving, reveals and makes use of the mathematical ideas that motivate it. In a similar vein, Arsac *et al.* (1992) proposed three roles for proofs as part of an instructional task: to understand why and/or to know, to decide the truth-value of a conjecture, and to convince oneself or someone else. Consequently, in the teaching experiment we adopted this broader view of the role of proof that goes beyond establishing the truth-value of a statement. In fact, we adopted a design principle according to which a proof has the potential to help students understand why a specific phenomenon happens. Therefore, in the algebra problems we focused on in the teaching experiment, the role of proof was not only to establish the truth-value of a statement but also to foster students' understanding of why a specific mathematical phenomenon happens.

Conjectures and Conjecturing Process

Another critical aspect of our work, following Boero (2007), is that conjecturing and proving are inter-related crucial mechanisms for generating mathematical knowledge. The notion of *cognitive unity* brings attention to the elements (e.g., mathematical relations, mathematical objects, etc) that could bridge the production of the conjecture with the production of its proof. In other words, some of the knowledge resulting from students' conjecturing process may be crucial at the moment of constructing the proof. Boero *et al.* (2007) systematically studied the relationship between conjecturing and proving in arithmetic and geometry with students in grade 8. However, we lack studies focusing on the relation between conjecturing and proving in the context of algebraic proof. In fact, this is still uncharted territory with the exception of Pedemonte's work (2008). Therefore in this paper, we focus on students' conjecturing process in the context of a teaching experiment aimed at providing students the opportunity to learn about algebraic proof.

In summary, in our teaching experiment, proof was conceived as an explanation accepted within the classroom community at a given time (Balacheff, 1988). The focus was on proof as an explanation that shows why a particular mathematical phenomenon happens (Arsac *et al.,* 1992; Hanna, 1990). In addition, building on Boero's work (2007), the mathematical tasks were not introduced as "proving tasks" or "proof problems." Instead, students were given an open-ended problem, and as a result of their exploration, produced a conjecture. Only after that, would the teacher prompt them to "show why this is always true."

METHODOLOGY

Participants

In this paper, we will describe and analyse the work of three $9^{th}/10^{th}$ grade students (Abbie, Desiree, and Grace) who participated in a teaching experiment, led by the first author of this paper, in which a total of nine $9^{th}/10^{th}$ graders took part, at a public charter school in the Boston area, Massachusetts, in the USA.

Procedure

Lessons: Fifteen one-hour lessons were held once a week. In this paper, we will report on data collected during Lesson 1, in which students focused on Problem 1, Part 1 (Figure 1). In this problem, students were not provided with the conjecture to prove. Moreover, it was presented as a standard math problem. Students were asked to analyse the nature of the outcome of the described calculation (subtraction of the cross product). It was expected that students would anticipate some kind of variation in the outcome in relation to the set of days where the operator is applied. However, students would find out, through exploration, that the same outcome is always obtained, no matter where they apply the operator. From a design point of view, it was intended that as part of the problem solving process, students would generate their own conjectures, and, at a later stage, would prove them.

Problem 1

Part 1: Consider a square of two by two formed by the days of a certain month, as shown below. For example, a square of two by two can be

1	2
8	9

These squares will be called 2x2 calendar squares. Calculate the difference between the products of the numbers in the extremes of the diagonals. Find the 2x2 calendar square that gives the biggest outcome. You may use any month of any year that you want.

Part 2: Show and explain why the outcome is going to be -7 always.

Figure 1. Problem 1 from the Algebra Calendar Sequence.

FINDINGS

Conjecturing Process

First, we will describe students' conjecturing process. Second, a definition of conjecturing will be proposed along with what is involved in the process and its potential products.

A map (Figure 2) of students' conjecturing process was produced taking into account different elements such as conjectures, order of appearance, and use of numeric examples. Students (i.e., Abbie, Grace and Desiree) produced several conjectures; each of them focusing on different elements that may impact the value of the outcome. They explored the possibility of each of the conjectures being true. In doing so, students studied the behaviour of the outcome and its dependence on/independence of different elements of the context.

Students produced a first conjecture, C1, linking the outcome and the placement of the square at the *beginning* of the month. After trying several cases, the conjecture was dismissed. The second conjecture, C2, is a claim about the non-dependence of the outcome in relation with the *month* where the square is placed. Students tried with different months, and, as a result of an arithmetical mistake a third conjecture, C3 was momentarily embraced. The possibility of the outcome's dependence on the month where the square is placed -in contradiction with C2-. After tyring with more examples, students dismissed C3 and produced a fourth conjecture. C4 relates the potential variation of the outcome according to the day of the week that is the first day of the month –for that month-. To test C4, students tried months that differ on the day of the week where the first day of the month falls. For all cases, the same outcome was obtained (i.e., -7). As a consequence, students produced their fifth and concluding conjecture (C5) stating that the outcome is always -7 regardless of the location of the square.

Thus, based on our empirical study, *conjecturing* is a complex process:

> that involves the production of several mathematical statements;
>
> from which, one of the conjectures emerges as a the conjecture to prove; and,
>
> through which a person comes to believe the likely truth-value of the conjecture to prove.

In other words, conjecturing is the process through which someone becomes confident about the plausibility of a certain conjecture. Indeed, *what* has to be proved is not obtained through a straightforward process, but rather through students' analysis of the likelihood of the truth-value of each of the produced conjectures.

In addition, the conjecturing process involves students' explicit or implicit dismissal of conjectures (i.e., C1 through C4) based on their exploration of the problem. Note that the use of examples is crucial at this stage as they inform students' decision regarding whether or not to dismiss a given conjecture.

Another significant product of this process is the production of mathematical relations that may be useful when constructing its proof. For instance, students identified that there is a difference of 8 and 6 between elements of the two diagonals (Figure 3). Later on, when these students were proving Conjecture 5 was central to their success.

In sum, the most prominent cognitive products steaming from this process are: dismissal of conjectures and the use of examples supporting their dismissal, identification of mathematical relations, plausible conjecture/s, and the underlying understanding as to why it is *plausible* for the chosen conjecture (C5) to be true.

Types of conjectures

In our analysis, we identified three distinct types of conjectures. *Competing conjectures* are all conjectures that were produced during the conjecturing process. As mentioned earlier, during the conjecturing process, students not only produced conjectures but also become certain of the likely truth-value of the mathematical statement. The resulting knowledge of the conjecturing process includes two types of conjectures. *Dismissed conjectures*, as it name indicates, are the conjectures that students dismissed implicitly or explicitly based on their exploration of the problem. *Conjecture to prove* is the conjecture that emerges from the conjecturing process as the plausible conjecture. Therefore, this is the conjecture that students set out to prove.

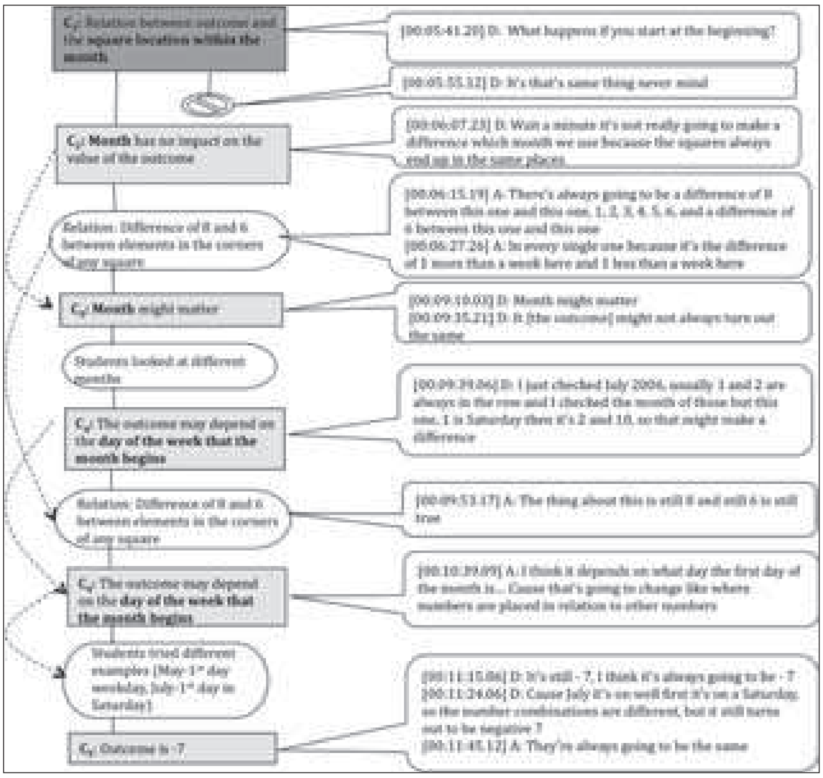

Figure 2. Conjecturing process.

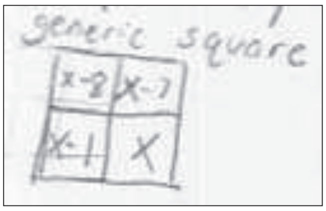

Figure 3. Desiree's representation of the relation identified during the conjecturing process linking elements in the diagonals and used for the proof.

Dismissed conjectures

These are the conjectures that are dismissed throughout the conjecturing process (i.e., C1-C4). However, they play a key role in producing the *conjecture to prove*. Based on our analysis, two different products resulting from the dismissal of a conjecture were identified.

On the one hand, the process that results in dismissing a conjecture may lead to a revised *dismissed conjecture* "closer" to the *conjecture to prove*. For example, in analysing C3 (i.e., month might matter) students tried different examples, namely they calculated the outcome corresponding to different moths. Students observed that doing this did not change the numeric value of the outcome. Therefore, they dismissed the third conjecture (C3) and produced a new revised conjecture (C4).

On the other hand, the process that results in dismissing a conjecture may lead also to the *conjecture to prove*. For example, as a result of dismissing the fourth conjecture C4, C5 is produced, which is the *conjecture to prove*.

As a result of dismissing each of the conjectures, students learn something about the problem and/or outcome. As a result of dismissing C1, students seem certain that the outcome does not depend on the square location. As a consequence of rejecting C3, students seem confident that the outcome does not depend on the month. The dismissal of C4 seems conducive for students to know that the value of the outcome does not depend on the day of the week that the month begins. The knowledge resulting from rejecting conjectures taken together with the fact that students kept obtaining the same numeric outcome (i.e., -7) led to the production of the conjecture to prove (C5).

Conjecture to prove

In this context, the *conjecture to prove* is knowledge about the problem produced by students that emerged as a result of the conjecturing process. Thus, the conjecturing process enables a meaningful construction of what must be proved. This stands in contrast to how proving activities are structured regularly in the US. Students are given *what* has to be proved without giving them a chance to explore the problem –a conjecture *is* a mathematical problem-, and, eventually, arrive to that conclusion themselves.

In summary, conjecturing is the process through which a person constructs a conjecture and becomes certain about its plausibility. In doing so, students learn about the problem, set up and study multiple scenarios; produce various conjectures and analyse their plausibility, and come to understand why some conjectures are dismissed. As part of this process, students also identified mathematical relations (i.e., difference between element in the diagonals) that are potentially useful in producing a proof. Thus, conjecturing offers many potential learning opportunities that our students may benefit from.

CONCLUDING REMARKS

In this paper, we have proposed a definition of *conjecturing* of value to the mathematics education community: is the process through which a person produces a mathematical statement and becomes *confident* about its *plausibility*. We illustrated this process with a map representing students' conjecturing process. As part of this process, students produce knowledge about the problem; this enables a meaningful construction of the conjecture to prove. In doing so, students also dismiss other competing conjectures based on their exploration of the problem. In this exploration, the purposeful selection of examples (and/or counterexamples) played a crucial role in both dismissing conjectures and reassuring them of the plausibility of the conjecture to prove. Another important product of this process is the construction of mathematical relations that are central to the construction of a proof.

By showing the cognitive value of having students construct their own conjectures, we hope to bring attention to core mathematical activities that we are depriving students in the traditional approach to proof in the US. Continuing to embrace the current curriculum, we are systematically taking away the opportunity to: produce their own conjectures, to reject non-plausible conjectures, to purposely select examples to either reject a conjecture or to reinforce the certainty on the plausible conjecture, and, last but not least, the construction of mathematical relations during the conjecturing phase that will be central to the construction of the proof.

References

Arsac, G., Chapiron, G., Colonna, A., Germain, G., Guichard, Y., & Mante, M. (1992). *Initiation au raisonnement déductif au collège*. Lyon, France: Presses Universitaires de Lyon.

Balacheff, N. (1988). Aspects of proof in pupils' practice of school mathematics. In D. Pimm (Ed.), *Mathematics, Teachers and Children*. London: Hodder & Stoughton.

Boero, P., Garuti, R., & Lemut, E. (2007). Approaching theorems in grade VIII: Some mental processes underlying producing and proving conjectures, and condition suitable to enhance them. In P. Boero (Ed.), *Theorems in School: From History, Epistemology and Cognition to Classroom Practice* (Vol. 1). Rotterdam, The Netherlands: Sense Publishers.

Boero, P., Garuti, R., & Mariotti, M. A. (1996). *Some dynamic mental processes underlying producing and proving conjectures.* Paper presented at the 20th Conference of the International Group for the Psychology of Mathematics Education, Valencia.

Garuti, R., Boero, P., & Lemut, E. (1998). *Cognitive unity of theorems and difficulties of proof* Paper presented at the 22nd Conference of the International Group for the Psychology of Mathematics Education Stellenbosh.

Hanna, G. (1990). Some pedagogical aspects of proof. *Interchange, 21*(1), 6-13.

Herbst, P. G. (2002). Establishing a Custom of Proving in American School Geometry: Evolution of the Two-Column Proof in the Early Twentieth Century. *Educational Studies in Mathematics, 49*(3), 283-312.

Pedemonte, B. (2008). Argumentation and algebraic proof. *ZDM, 40*(3), 385-400.