DISCOVERIES AND (EMPIRICAL) VERIFICATIONS – OPTIONS FOR TASK DESIGN

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Mathematical theorems can be discovered and verified in different ways. In this paper, I present a system of options, in which the plurality of ways of constructing and establishing theorems throughout mathematical tasks is reduced to a few basic elements. The framework is based on considerations of the philosophical logic of Charles S. Peirce: abduction, deduction and induction. The options are exemplified by problems, which challenge students to discover and verify mathematical theorems. The pros and cons of the options are going to be discussed.

INTRODUCTION

Since the raise of constructivist approaches (e.g., Cobb et. al., 1992) mathematics education research has focused on individual and social learning processes. In spite of the importance of these processes the theoretical background of setting up hypotheses has been considered in the recent years: the abduction. Abduction has been used theoretically to present the necessary inference for the acquisition of knowledge (e.g., Voigt, 1984; Hoffmann, 1999). Also there are first approaches to reconstruct abductions from students' comments (e.g., Knipping, 2003; Pedemonte, 2007). Meyer (2007) established an alternative pattern of abduction based on the later theory of Peirce and refined by reconstructing classroom communication.

In this paper I will describe how the theory of Peirce can be helpful in order to analyse problems in mathematical school books which challenge students to discover and verify mathematical theorems. I am going to use the three inferences (abduction, deduction and induction) to present a system of options, in which the plurality of ways of constructing and establishing theorems is reduced to a few basic elements.

ABDUCTION, INDUCTION AND THEIR INTERPLAY

In this chapter, the inferences abduction and induction will be considered. Mainly I will discuss the benefits of Peirce's later concept of abduction for the understanding of discovering new mathematical theorems.

Abduction

In the course of his philosophy, Peirce offered several different descriptions and patterns of abduction. In his later writings he defines the "perfectly definitive logical form" (Peirce, CP 5.189) of abduction as follows:

> "The surprising fact, C, is observed;
> But if A were true, C would be a matter of course,
> Hence, there is reason to suspect that A is true." (CP 5.189, 1903)

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Hempel und Oppenheim claimed that an explanation has to contain at least one general rule (Peirce also used a general rule in his former writings, cf. CP 2.622). Furthermore the observation has to be deductively inferable from the explanation (cf. Stegmüller 1976, p. 452). Thus we get a rule as mediator between the observed fact (C) and the case (A).

fact (result):	$R(x_0)$	result:	$R(x_0)$
rule:	$\forall i: C(x_i) \Rightarrow R(x_i)$	rule:	$\forall i: C(x_i) \Rightarrow R(x_i)$
case:	$C(x_0)$	case:	$C(x_0)$

Figure 1: Patterns of abduction

If we take a closer look at the pattern of abduction, we are confronted with a problem: The case is entirely contained in the rule. If the rule is present, the case is present, too. The rule can not be a premise. Thus we have to differentiate between abduction as a process of presenting our hypothesis plausibly to public (right in figure 1) and abduction considered as a cognitive process (left in figure 1). The latter starts with only one given premise: the observed fact. This fact occurs as result of a general rule, if we are aware of this rule.

The cognitive process of finding an explanatory case is not logic in the sense of mathematical formal logic. We rather see ourselves confronted only with a surprising fact, before we carry out an abduction. We take the fact for granted, irrespective of what we think about it. We feel constrained to accept it. It is the fact per se but for us the fact is only present against the background of our own cognitive abilities. If we are aware of the rule, the observed fact appears as a result of this rule. For this, we put ideas of our former knowledge together and create something new, which is – given by the observed fact – supported by reality (cf. CP 5.181).

The differentiation between abduction as a cognitive and a public process implies that not only new cases but also new rules (and on this base new theories) can be inferred abductively: If the observed fact could not be explained by using known rules, a new rule (and thus a new case) has to be created. Eco (1983, pp. 206) calls this a "creative abduction". Thus, if a student has to discover a new rule, the task has to provide results of the rule the student can not explain by known rules.

Induction

Induction is the inference from a case and a result to a rule. Figure 2 shows the general pattern of an induction. In the common point of view, induction is the necessary inference for the creation of new rules. The underlying concept of this point of view can be described as follows: 'What can be observed a couple of times is always valid'. If we take a closer look at the pattern of induction, we are confronted with a problem: The inference starts with the combination of a fact and a result, but how do we get the idea of combining these premises? This can only be done by abduction. Induction only takes place in order to confirm new ideas.

case:	$C(x_0)$
result:	$R(x_0)$
rule:	$\forall i: C(x_i) \Rightarrow R(x_i)$

Figure 2: The pattern of induction

The interplay of the inferences

In his later writings Peirce does not consider induction as being the inference to create new rules: “in almost everything I printed before the beginning of this century I more or less mixed up Hypothesis and Induction” (Peirce, CP 2.227; afterwards he called hypothesis abduction). Induction now appears as the inference from ideas to the confirmation of these ideas and abduction appears as the inference from observed facts to new ideas. Thus his new conceptualisation differs from the common concept of induction and he defines the inferences as different steps in the process of inquiry:

> “… there are but three elementary kinds of reasoning. The first, which I call abduction … consists in examining a mass of facts and in allowing these facts to suggest a theory. In this way we gain new ideas; but there is no force in the reasoning. The second kind of reasoning is deduction, or necessary reasoning. It is applicable only to an ideal state of things, or to a state of things in so far as it may conform to an ideal. It merely gives a new aspect to the premisses. … The third way of reasoning is induction, or experimental research. Its procedure is this. Abduction having suggested a theory, we employ deduction to deduce from that ideal theory a promiscuous variety of consequences to the effect that if we perform certain acts, we shall find ourselves confronted with certain experiences. We then proceed to try these experiments, and if the predictions of the theory are verified, we have a proportionate confidence that the experiments that remain to be tried will confirm the theory. I say that these three are the only elementary modes of reasoning there are.” (CP 8.209)

1. Abduction	2. Deduction	3. Induction
result 1	case 2	case 2
rule 1	rule 1	result 2
case 1	result 2	rule 1
1. Abduction	2. Deduction	

Figure 3: The structure of verification by the Bootstrap-Model (the same rule in every step)

There are at least two ways how this logical combination of abduction, deduction and induction can be used for the confirmation of a hypothesis: The Bootstrap-Model (Carrier, 2000, p. 44) points out that the subjects $(x_{0,\dots,i})$, which motivated the creation or association of the general rule, are specific. Other similar subjects $(x_{i+1,\dots,m})$ can be used to confirm or to refute the rule or its coherence to the result (figure 3). Starting with some observed facts (result 1) we abductively suggest a general rule. In the second step this rule is used to deduce a prediction: If the rule is correct, a similar case (case 2) must have a necessary consequence (result of the deduction). The predicted result is now going to be tested. If the test confirms the prediction, the rule gets confirmed by another example.

1. Abduction	2. Deduction	
result 1	case 2	(= rule 1 or case 1)
rule 1	rule 2	
case 1	result 2	

Figure 4: The structure of hypothetico-deductive verification (without the inductive step)

A second approach of verification uses another rule, the hypothetico-deductive approach (Carrier, 2000, p. 44): Deductively another rule is used for inferring a consequence of the conjectured rule or case. If the rule or the case of the abduction is correct, the prediction (the result of the deduction) has to follow. If the test does not confirm the prediction, the induction proves the hypothesis to be wrong.

OPTIONS OF TASK DESIGN

The described distinction between the inferences and their interplay for the empirical confirmation of discovered rules has been used to analyse tasks in textbooks. In this chapter I will describe the way and the outcomes of the analyses.

Methodology

The characteristic inference for discovering mathematical theorems is the abduction. An abduction gets initiated by only one given premise: the surprising facts which appear as a result of a rule, if we are aware of this rule. If a task in a textbook is supposed to lead the students to a theorem, the task has to provide concrete results of this theorem or (at least) the task has to enable the students to gather them. This implies the following consequences for the reconstruction of tasks (The method of reconstruction is described shortly in this paper. For a more detailed characterisation please consider Meyer and Voigt, 2009): First, the scientist can look for mathematical theorems the students should discover. These rules are often described in the following course of the book or in the reference book for the teacher. Second, the task has to be considered in different ways in order to establish a result of the theorem. Sometimes these results are presented directly in the task. If there is no mathematical theorem explicitly mentioned in the reference book or in the textbook, a task could lead to different theorems. Thus, sometimes different ways of discovering could be reconstructed. The reconstruction of analysing different ways of empirical verifications of a mathematical theorem can be done analogically. Examples will be presented in the following course of this paper. In the concerning project, about 50 schoolbooks have been analysed. For the reconstruction of different latent meanings of possible student solutions the method of "objective hermeneutics" (Oevermann et al., 1979) has been used.

By analysing the tasks, different logical structures for discovering and verifying theorems could be observed and resulted in a system of options. These options are exemplified in consideration of the following theorem:

When multiplying two powers with the same basis a^b and a^c ($a,b,c \in$ [illegible]), then the product $a^b \cdot a^c$ equals a^{b+c}.

Figure 5: Multiplication law for powers with the same basis

Options for task-design I: Discovering mathematical theorems

Option 1: Discovery by a special result. In order to discover the multiplication law a student could rewrite 100 000 · 100 = 10 000 000 by using powers with the basis 10. Therefore, he only has to use the definition of powers deductively. The new equation can be used in order to discover a general rule. The discovery of the multiplication law can be made by the following abduction:

result:	$10^5 \cdot 10^2 = 10^7$ (by using the law of attaching zeros)
rule:	Multiplication law
case:	The bases are the same and $5+2=7$.

Figure 6: A creative abduction made by a special result.

Concerning the example, the result of the abduction might be so familiar for a student that the law of attaching zeros could be enough for him to explain the result. We speak of a "discovery by a special result", if known rules can be used to explain the result and the student is not in need of finding a new rule.

Option 2: Discovery by a typical result. The abduction by a typical result increases the chance to discover a new theorem (instead of using known theorems). Regarding the example: The student determines $3^5 \cdot 3^2$ and 3^7 by using a calculator. Afterwards he should – in consideration of the equality – discover a general rule. The necessary abduction is nearly analogous to the former one. The result is a "typical result" for the theorem, because the students have yet not been in contact with rules, which compete with the requested theorem in order to explain the result abductively.

Option 3: Discovery by a couple of results. A mathematical theorem can get more plausibility, if it is discovered by more than one result. The risk decreases that the theorem might only be valid for particular kinds of numbers. The students, for example, calculate $243 \cdot 9$, $16 \cdot 4$ and $25 \cdot 5$ and determine the outcomes by using powers with the lowest basis. Contrarily to figure 6 the quantity of the equitations and the number would change in the result and the case of the abduction.

Option 4: Discovery by a class of results. If the results form a whole class, the discovery of a theorem can get a particular plausibility. Concerning the example, the students could be required to remember the rule for attaching zeros in order to multiply powers with the base 10. They should write down this (known) rule by using powers with the base 10 and recognise a more general (!) rule. The following abduction describes this option:

result:	$10^r \cdot 10^s = 10^{r+s}$ (by using the law of attaching zeros)
rule:	Multiplication law
case:	The bases of the multiplied powers are the same.

Figure 7: A creative abduction made by a class of results.

Option 5: Discovery with a latent idea of proof. This option does not only lead to a mathematical theorem, but also to an idea of proof of this theorem. In order to discover the multiplication law with a latent idea of proof, the following tasks could be solved: Firstly, write 3^5 and 3^2 as products ($3^5 = 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3$ and $3^2 = 3 \cdot 3$). Secondly, form the product of these products and write it down by using powers with the basis 3 [$3^5 \cdot 3^2 = (3\cdot3\cdot3\cdot3\cdot3)\cdot(3\cdot3) = 3\cdot3\cdot3\cdot3\cdot3\cdot3\cdot3 = 3^7$]. Thirdly, discover a general rule.

result:	$3^5 \cdot 3^2 = (3\cdot3\cdot3\cdot3\cdot3)\cdot(3\cdot3) = 3\cdot3\cdot3\cdot3\cdot3\cdot3\cdot3 = 3^7$
rule:	Multiplication law
case:	The bases of the multiplied powers are the same and $5+2=7$.

Figure 8: Discovery with a latent idea of proof

The result of the abduction is getting inferred by a chain of deductions. If these deductions are generalised, it is possible to prove the theorem (the rule of this abduction). In other words: In order to elaborate the result the students are confronted with the main steps of the proof of the theorem before they become aware of it. The idea of proof is latent because the students have to recognise it and this is not self-evident.

Options for task-design II: Verifying mathematical theorems empirically

Option 1: Verifying by a special case. After the student has discovered the multiplication law by one or more results an empirical verification is in need of a new case. Let us consider the example the theorem has been discovered by calculating $3^5 \cdot 3^2$ and 3^7. Now the students are asked to verify it with $3^6 \cdot 3^7$ by using the calculator. If the verification is successful the following induction confirms the theorem.

case:	In $3^6 \cdot 3^7$ the bases are the same and $6+7=13$.
result:	$3^6 \cdot 3^7 = 3^{13}$ (by the calculator)
rule:	Multiplication law

Figure 9: An induction made by a special case

The problem of this way of verification is that the basis 3 has already been used to discover the multiplication law. Thus, the students could recognise a general rule which is only valid for the basis 3. Anyway, the verification does not confirm the theorem as being a general one.

Option 2: Verification by a specific other case. If the verification has to be done by numbers which are completely different to those which had been used to discover the theorem, the verification gets a specific plausibility. Thus, the fear decreases of verifying a theorem which is only valid for a few or a particular number.

Option 3 and 4: Verification by a couple of cases (3) and by a class of cases (4). If the number of the cases of a theorem is increased (and not always with the same basis), the theorem can get a lot of plausibility:

> "[...] having verified the theorem in several particular cases, we gathered strong inductive evidence for it. […] Without such confidence we would have scarcely found the courage to undertake the proof which did not look at all a routine job." (Polya, 1954, pp. 83)

If the theorem should be verified by a class of cases, the student needs known rules to infer the result of the induction. Considering the example, he could use the law of attaching zeros. Figure 10 shows an induction for using this option:

case:	In $10^b \cdot 10^c$ the bases of the multiplied powers are the same.
result:	$10^b \cdot 10^c = 10^{b+c}$ (by using the law of attaching zeros)
rule:	Multiplication law

Figure 10: An induction made by a class of cases

Option 5: Verification with a latent idea of proof. Nearly analogous to the option 5 for discovering mathematical theorems, the verification of a theorem can also imply a latent idea of proof. Now the deductive steps, which can be used structurally for the proof, need to be done to infer the result of the deduction. Let us assume the students had discovered the theorem by the result $3^5 \cdot 3^2 = 3^7$ which had been determined by the calculator. Now they should verify it by $7^4 \cdot 7^5$ without using the calculator. The amount of the numbers causes the students not to solve the task by calculating.

Option 6: Verifying by networking. The former options for verifying theorems depend on the Bootstrap-Model: A theorem gets more plausibility, if we check more examples for it. The following option bases on the hypothetic-deductive approach. Unfortunately, this option could only be reconstructed a few times in textbooks. Let us consider an example of this option: The student assumes the multiplication law to be correct. Now he infers that this theorem can also be verified by geometry. In geometry, powers can be described by areas and volumes. In the three-dimensional space, a successful verification of the multiplication law can happen, if the student realises that the volume of a cube can be determined by “base area times height”:

case:	a^1 is the height of a cube and a^2 the base area. In terms of height and base area, a^1 and a^2, the bases of the powers are the same, and $1+2=3$.
result:	$a^1 \cdot a^2 = a^3$ is the volume of a cube.
rule:	Multiplication law

Figure 11: Verifying by networking

FINAL REMARKS

An abduction starts with only one given premise and cannot guarantee certainty. The Bootstrap-Model and the hypothetic-deductive approach show how abductively inferred knowledge can be verified. Both approaches are empirical ones. They can be used to confirm the theorem, but not to prove it. The analyses of textbooks resulted in different options for tasks-design in order to discover or to verify a theorem. The presented options for discovering mathematical theorems show how tasks can be created to enable students to recognise a theorem by a creative abduction that it is able to

- increase the chance to discover a new theorem,
- increase the plausibility of the discovered theorem and
- enable the students to recognise the idea of proof for the theorem while discovering the theorem.

The presented options for verifying mathematical theorems show how tasks can be created to verify an already discovered theorem that it is able to

- increase the plausibility of the discovered theorem,
- enable the students to recognise the idea of proof for the theorem while verifying the theorem and
- make students aware that a possibly hard deductive proof can be successful.

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