CONCEPTUALIZATION OF THE RELATIONS BETWEEN GEOMETRIC PROOF AND GEOMETRIC CALCULATION

Hui-Yu Hsu

University of Michigan, U.S.A.

This study aims to conceptualize the relations between geometric proof (GP) and geometric calculation (GC) through a survey study with the hypothesis: geometric diagram and geometric properties necessary for a solution are the keys to determine the relations between both types of tasks. The empirical analysis shows that students performed no significant differences between both types of tasks when they formally learned proof lessons in schools. But students performed significantly better on GC than on GP items when they have not learned proof constructions. According to the findings, implications of learning trajectory through GC to GP are provided.

INTRODUCTION

The value of teaching and learning proofs at the secondary school level has been a matter of some disagreement in the field. On the one hand, some scholars have argued that proofs and reasoning are fundamental to knowing and using mathematics (Ball & Bass, 2003). In line with this view, NCTM (2000) identifies proof and reasoning as a topic that should be taught across all grade levels, thus making the teaching proof as a central goal of mathematics education. On the other hand, teachers and students often resist proofs, and, as a result, proofs have been relegated to a less important role in curriculum, because of the teaching and learning difficulties encountered in the countries with a tradition of teaching proof in secondary curriculum (Mariotti, 2006). In geometry, when proofs are regarded as a difficult topic for students, geometric calculations are an attractive alternative to formal proofs (Schumann & Green, 2000). The reason for this is because geometric calculations can provide opportunities for students to become familiar with and apply geometric properties, the major aim in the secondary geometry curriculum. Another reason comes from the perspective of cognitive development. Using geometric calculations to practice geometric properties is aligned with children's development of geometry and space conceptions. As Piaget, Inhelder, and Szeminska (1960) indicate, children at the elementary school level are capable of performing calculating tasks by applying geometric properties (e.g., size of angle, length of distance) and using calculations to envisage the truth of mathematics. Theories have been proposed to specify the relation between proofs and calculations in general (Tall, 2002), however, these theories do not deal specifically with the relations between geometric calculations and geometric proofs, especially how geometric diagrams may possibly influence the relations. Some empirical studies have attempted to survey students' performance on geometric calculations and geometric proofs (Healy & Hoyles, 1998). However, they could not articulate the relations between both types of tasks

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because of the diverse factors that may influence the comparisons (e.g., the geometric properties required to obtain the solutions). Another way to assess the relations is to directly view geometric calculations as the tasks only for applying properties and rules (Heinze, Cheng, & Yang, 2004; Heinze, Ufer, Cheng, & Lin, 2008). But, the way to directly treat geometric calculations as lower cognitive-demand tasks may take risks in elaborating the relations because they ignored the complexity and particularity a task can be (Stein, Smith, Henningsen, & Silver, 2000). To conceptualize the relations between GP and GC, this study hypothesises that the decisive factor to the relation is students' ability to visualize the diagram in such a way that allows them to retrieve the geometric properties necessary for a solution. Thus, a survey study was implemented to examine the proposed hypothesis.

LITERATURE REVIEW

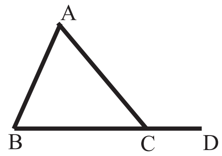
A diagram can be viewed as a strategic location where problem solving happens (Larkin & Simon, 1987), and they can serve as schemes by which students remember the steps in solving a problem, the given statements, and the diagram labels (Lovett & Anderson, 1994). Diagrams can be parsed into chunks to cue the geometric knowledge needed for solutions, which mirrors how experts solve geometric tasks (Koedinger & Anderson, 1990). Moreover, diagrams can also function as an artifact in scaffolding students in learning proofs. Cheng and Lin (2006) reported that junior high school students' performance on constructing proofs were improved greatly through instructional intervention by asking students themselves to read the given information and then color these properties on the diagrams. The colored parts of the diagram help students visualize the useful geometric properties for forming an acceptable proof. Furthermore, Fujita, Jones, and Yamamoto (2004) also claim that creating and manipulating geometric diagrams mentally or physically can nurture students' intuition in geometric problem solving. In this sense, diagrams can bring geometric calculations much closer to geometric proofs and can be the key to influencing students' performance on the two types of tasks. Another relevant focus of the geometric diagram is the requirement of geometric properties. No matter if they are solving geometric calculations or doing geometric proofs, students need to visualize the geometric diagrams and identify the needed geometric properties in order to set up calculating sentences or form logical proving statements. Research has reported such work of visualizing the diagram and identifying properties geometric properties to obtain solutions is difficult for students (Fischbein & Nachlieli, 1998; Zykova, 1975).

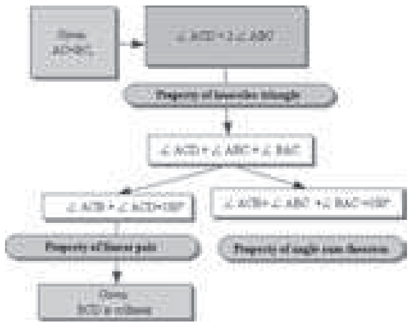
METHODOLOGY

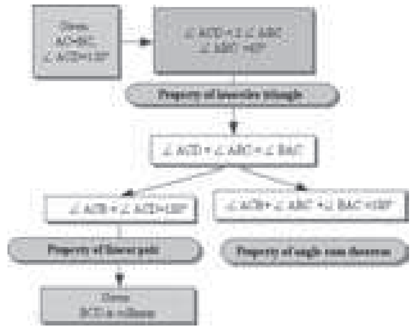
Survey Design

A survey consisting of 4 pairs of GC and GP items was designed based on five principles. The first principle is to control the diagram configurations used by the items. Pairs of GC and GP items employ the same diagram configurations, in which the labels are located on the same places and the size and orientation of the diagrams in the survey are all identical. The second principle has to do with the requirement of the geometric properties necessary for a solution. While a task may have more than one solution and may require different sets of geometric properties for solutions, the identical diagrams provide students the same opportunities to visualize and retrieve corresponding geometric properties. This results in the requirements of geometric properties being the same for the pairs of GC and GP items. As the below pair of GP and GC, two sets of geometric properties can generate solution plans for the pair.

The first set of geometric properties is the triangle sum property, the property of linear pair, and the isosceles triangle property. The second set of geometric properties is the exterior angle property and the isosceles triangle property. The third principle is to control the sequence of corresponding geometric properties necessary for solutions. Again, when the given diagrams are identical, the study also can control when and how the geometric properties are needed in a solution plan for pairs of GC and GP items.

Given diagram	GP item	GC item
	BCD is collinear and AC = BC. Prove ∠ACD=2∠ABC	BCD is collinear and AC＝BC. If the measure of ∠ACD is 130°, find the measure of ∠ABC.





The figures above show the solution stages for GP (the left one) and GC items (the right one), which indicate that needed geometric properties appear in the same sequence for both items. Students need to apply the property of linear pairs and the triangle sum property to obtain the statement that ∠ACD=∠ABC+∠BAC. Next, students can apply the isosceles triangle property to obtain the GC answer which is 65°and prove the statement ∠ACD=2∠ABC. Parallelizing the required geometric properties for solutions also ensures that student's attention is directed to the same configuration of the diagram, thus, imposing the same cognitive demand. The fourth principle is to vary individual pairs of items in terms of the shape of geometric diagrams and the requirements of geometric properties necessary for a solution. Thus, this study can prevent superficial learning from students' prior experiences of working on a similar diagram setting, which later may become a confounding variable for the comparison on GC and GP items.

Survey Procedure and Sample

This study employed two survey procedures to administer the items to students. As it is unclear how students experience doing GP items may influence their performance on the paired GC items and vice versa, using different survey procedures can benefit to the understanding of the influence of working on GC items before or after working on GP items. Using the previous pair items, students can calculate the answer for GC item directly ($\angle ABC=65^\circ$) by applying the conclusion statement of its paired GP ($\angle ACD=2\angle ABC$). The direct application of the GP conclusion may free students from making inferences based on the given information (e.g., AC=BC) or the geometric properties (e.g., exterior angle property), which is necessary for a solution that doing paired GP requires. It is also possible that the GP conclusion statement also provide student a hint to the paired GC solution because they can reason backward from the conclusion ($\angle ACD=2\angle ABC$) to generate a solution plan of GC. It is also unclear whether or not asking students to solve GC and GP items at different times influences their performance on the two types of items. To further understand how students' work on both types of tasks interacts as they work on each other, this study creates a four-condition model. The main idea of the model is to ask students to work on GP and GC items in different order and different time sequences.

Survey Condition	Day 1	Day 2
Condition 1: GP first and GC later	4 GP	4 GC
Condition 2: GC first and GP later	4 GC	4 GP
Condition 3: GP first and GC later	2 GP first and 2 paired GC later	2 GP first and 2 paired GC later
Condition 4: GC first and GP later	2 GC first and 2 paired GP later	2 GC first and 2 paired GP later

All participating students completed survey items on two consecutive days. Condition 1 referred to the situation in which students solved all four GP items on Day 1 and solved paired GC items on Day 2. Condition 2 was opposite to Condition 1 and asked students to solve GC items on Day 1 and paired GP items on Day 2.

For Condition 3, students were required to solve two GP items first and then two paired GC items on Day 1. These students worked on another two pairs of items in the same way on Day 2. Condition 4 was also opposite to Condition 3 and asked students to work on two GC items first and then two paired GP items on Day 1 as well as the other two pairs of GC and GP items in the same sequence on Day 2. Because of the study design, Condition 1 and Condition 2 allowed a direct comparison of order effects (GP before or after GC) as well as Condition 3 and Condition. In addition, conditions 1 and 3, as well as conditions 2 and 4, should also allow a direct comparison of timing effects (i.e., paired GP/GC items solved close in terms of time versus separated by one day). No students were informed that they needed to work on GP or GC items on Day 2. Students were also not allowed to revise their previous work while they worked on the paired items. A total of 413 9^{th} grade students and 502 8^{th} grade students from three middle schools in Taiwan constituted the valid sample for testing the hypothesis. By selecting these two grades of students, this study could also compare different experiences of learning geometric proofs in their performance on GP and GC respectively. The Taiwanese curriculum standard of mathematics uses the geometric content related to triangle congruence and parallel lines properties to gradually introduce the proof concept to students in 8^{th} grade. In this stage, students are only required to fill out one of the steps in a proof or provide the corresponding geometric properties for a proof step. For 9^{th} grade, geometric proofs are formally introduced and students are required to learn how to construct proofs themselves. The surveys were administrated to 9^{th} graders after they had completed all the proof lessons and to 8^{th} graders after they had learned the geometric knowledge that are necessary for answering the pairs of survey items. In order to prevent the disparity of classes and schools from being a confounding variable in the comparison of students' responses, student participants were assigned to condition treatments on the class basis. That means students in each class were equally and randomly assigned to one of the four condition treatments and completed the required survey items in different time sequences on two consecutive days.

Coding

Cheng and Lin (Cheng & Lin, 2005) clarified students' responses on geometric proofs into four levels based on crucial geometric properties of proof solutions: acceptable proof, incomplete proof, improper proof, and intuitive responses. Their clarification of these geometric properties in relation to proof solution echoes the hypothesis of the study, which assumes that the required geometric properties embedded in diagrams can determine student performance on GC and GP. Moreover, in order to compare pairs of both items, levels on the GC items corresponding to that of GP were established in the following manner: correct calculation with reasons, incomplete calculation, improper calculation, and intuitive response. Points were assigned for both GP and GC coding. Three points were assigned when students constructed an acceptable proof or provided a correct calculation with reasons. Two points were given when students' responses were coded as an incomplete proof or incomplete calculation. One point was assigned to students whose responses were coded as "improper proofs or improper calculations". Students who provided intuitive responses or gave up on the survey items received no points. For statistical purposes, students' points on the four pairs of items for GP and GC respectively were added. Moreover, students' error responses specific to GP items were also analysed. The reason for doing so is to understand what particular difficulties students may have in constructing proofs but these difficulties did not exist in GC responses.

FINDINGS

Students' performance on GP and GC items

The result shows that 9th graders performed no significant differences between GP and GC items (t= -1.945; p>.05), which means that the given diagram and the geometric properties needed for solutions determine the cognitive demand on 9th graders rather than the different formats between a proof and a calculation task. However, 8th grade students performed significant better on GC than GP items (t=-4.289, p<.05).

Comparison between two survey procedures

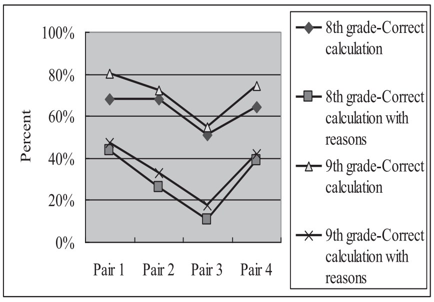
Statistical comparison shows that the survey procedures did significant influence students' performance on GC items. Students performed better when solving GP first and GC later than when solving GC first and GP later (t=2.156, p<.05). For GP items, survey procedures did not cause a significant difference (t= -.339, p>.05).

Comparison of the time effect

For the examination of the time effect on student performance for both types of items, the statistical results show that students' performance on GC for the two conditions (GC/GP solved closely together vs. separated by one day) is not significant (t=-.020, p>.984). A similar result was obtained on GP items (t= -.508, p>.05).

Comparison between correct calculation and correct calculation with reasons

This study further examines how superficial visual association of the diagram(Aleven, Koedinger, Sinclair, & Synder, 1998) plays a role in influencing students to obtain the numerical answer without understanding the geometric content. To achieve this goal, this study investigates students' responses between correct calculation and correct calculation with reasons. The table on the right side shows that the percentages of obtaining correct answers only are much higher than those of correct answers with reasons for both grade levels.



DISCUSSION

Based on students' responses on the survey items, this study conceptualizes the relationship between GC and GP taking the considerations into two perspectives. The first perspective is about the role of geometric diagram and geometric properties necessary for a solution. According to 9th grade students' responses, this study shows that diagram and the required geometric properties are the keys to determine cognitive demand imposing on students. This result also points out the weakness of theories and studies describing the relationship between calculation and proof (Heinze *et al.*, 2008; Tall, 2002) because these theories seem not consider how the diagram can play a role in influencing the relation between geometric calculation and geometric proof. The second perspective is about the format differences between a calculation and a proof task. The analysis also indicates that GP is more cognitively demanding than GC when students are not equipped with the proving skills. Further analysis of error responses specific to GP items indicates that the lower performance on GP may due to students' difficulties in (1) chaining the proving sentences into a logic order; and (2) properly using algebraic representations and geometric representations (e.g., ⊥ for right angle) to express a proving sentence. These two difficulties were not found in their responses on GC since each calculating step usually results in a numerical answer which will become the given information for the inference of next calculating step. The circulation of finding unknown measures and using just found measures to infer other unknown measures in a way scaffold students in chaining their solution in a reasonable sequence. Regarding the algebraic and geometric representations, setting up a calculating sentence requires much less demand than a proving sentence.

Piaget, Inhelder, and Szeminska (1960) claim that children at elementary school level are capable of performing calculations tasks by applying geometric properties. In line with their claim, the findings take further step to propose a learning trajectory which suggests students first to practice GC by visualizing the diagram configurations with relevant geometric properties. Later on, when they are equipped with the skills of proof constructions (e.g., chaining proving steps into a logic sequence), they can successfully transfer experiences of practicing GC tasks into GP construction. While research has reported students' difficulties in learning proofs (Fuys, Geddes, & Tischler, 1988; Heinze *et al.*, 2004), the learning trajectory may also be used to develop instructional strategies concerning the diagram and requirements of geometric properties to improve students' learning of GP, which is worth of further investigation.

In addition, the result that the percentages of obtaining correct answers only for GC tasks are much higher than that of correct answers with reasons echoes the shallow learning proposed by Alven *et al.* (1998). But the following research question should be how the shallow learning may influence students' competence of constructing geometric proofs.

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