A MODEL TO ANALYSE ARGUMENTATIONS SUPPORTING IMPOSSIBILITIES IN MATHEMATICS

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Starting from Toulmin's analysis of impossibility and through the study of some cases, I investigate argumentations supporting mathematical impossibilities. In particular, I discuss issues related to the notions of impossibility, as the contradictoriness, the criteria and the implications of an impossibility, and the Toulmin' pattern of argumentations produced to state that something is impossible. I show the use of this model to analyse students' argumentations, to identify the specific aspects of these argumentations, to describe the differences and the analogies between argumentations and proofs that support impossibilities.

INTRODUCTION

In the last decades, many research studies have been published on argumentation and proof in mathematics education (see Mariotti, 2006). In some studies (see, for example, Duval, 1992-93), a distance between argumentation and proof is claimed, while in others, the focus is put on the analogies between them, in particular between the processes involved in their constructions (Garuti *et al.*, 1996). In any case, researchers share the importance of investigating students' argumentations and, in many articles, some models to analyse argumentations are proposed. The most appreciated model is that of Toulmin (1958), a scheme with six components (data, claim, warrants, qualifiers, rebuttal, backing) which describes a step of argumentation. The articles by Krummheuer (1995), Yackel (2001), Hoyles and Küchemann (2002), Pedemonte (2007), Inglis *et al.* (2007), Knipping (2008) are only some of many papers in literature that refer to this model.

In this paper, I investigate argumentations that support an impossibility. This is a very common situation in mathematics and in classroom, when one has to conjecture, argument or prove that some object does not exist or that it cannot have some properties, or the impossibility of making some ruler and compass constructions, of solving an equation, etc. I will present a model to analyse these argumentations in different situations, to identify their general and specific aspects, to describe the differences and the analogies between these argumentations and mathematical proofs.

THEORETICAL FRAMEWORK

To analyse students' argumentations supporting impossibilities, I consider a wide and general notion of impossibility. We need this for two reasons. First of all, it seems reasonable to hypothesize that students can refer to extra-mathematical meanings of impossibility and support it as we usually do in non scientific fields. In addition, also

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mathematicians could treat impossibility with different criteria according to the functions of the argumentation they generate. In fact, following De Villers (1990), Winicki-Landman (2007) suggests that these functions (*verification, explanation, systematisation, discovery, communication*) assume an important role also in proving – and then in argumenting - an impossibility.

Even if the generality of the well known Toulmin's model (Toulmin, 1958) with six components is one of the more appreciated characteristics, for our goals it is a limit, because it makes difficult to identify some specific and important aspects of impossibility. On the other hand, Toulmin himself (1958) proposed an analysis of impossibility and a specific model of argumentation supporting impossibility that have been not considered in mathematics education. This part of his work is summarized in the following.

One of the main problems studied in (Toulmin, 1958) is that of distinguishing the features of arguments which are *field-invariant* from those that are *field-dependent*, where the *fields* of argumentations are, for example, Meteorology, Physics, Jurisprudence, Mathematics, etc. His first examples regard impossibility, in particular the structure of argumentation, the notion and the criteria of impossibility (Toulmin, 1958, 21-33). First of all, Toulmin proposes a field-invariant pattern of argumentation that support impossibility: "*P being what it is, you must rule out anything involving Q: to do otherwise would be R, and would invite S*" (Toulmin, 1958, p. 28). Toulmin offers different examples to show that he is considering a general notion of impossibility and a general pattern of argumentation supporting an impossibility:

> "Your physique being what it is, you can't lift that weight single-handed – to attempt to do so would be vain" (p. 23); "the nomenclature of sexes and relationships being what it is, you can't have a male sister – even to talk of one is unintelligible" (p. 25); "the bye-laws being as they are, you can't smoke in this compartment, Sir – to do so would be a contravention of them and/or an offence against your fellow passengers" (p. 27); "standing in the relationship you do to this lad, you can't turn him away without a shilling – to do so would be unfatherly and wrong" (Toulmin, 1958, p. 27).

If this pattern of argumentation of impossibility and the implication for something to be impossible are field-invariant, the criteria of impossibility are field-dependent:

> "The meaning of a modal term, such as 'cannot', has two aspects: these can be referred to as the *force* of the term and the *criteria* for its use. By the 'force' of a modal term I mean the practical implications of its use: the force of the term 'cannot' includes, for instance, the implied general injunction that something-or-other has to be rules out in this-or-that way and for such-a-reason. This force can be contrasted with the criteria, standards, grounds and reasons, by reference to which we decide in any context that the use of a particular modal term is appropriate. We are entitled to say that some possibility has to be ruled out only if we can produce grounds or reasons to justify this claim, and under the term 'criteria' can be included the many sorts of things we have then to produce." (Toulmin, 1958, pp. 28-29)

In summary, an impossibility is something that requires a reason, an argument grounded on criteria of impossibility that depend on the field, but it is also something which has implications: what is impossible has to be dismissed from consideration. Similarly, "*in order for a suggestion to be a 'possibility' in any context [...], it must 'have what it takes' in order to be entitled to genuine consideration in that context*" (Toulmin, 1958, p. 34)

The distinction between impossibility and contradictoriness is considered by Toulmin with particular attention to Mathematics. He reminds that, for example, mathematicians consider impossible that a number is both rational and a square root of 2 because the notion of 'a rational square root of 2' implies a contradiction (Toulmin, 1958, p. 29). To prove that a notion leads to a contradiction is required to say that this notion is impossible. In general, a mathematical notion is impossible on account of the contradictions. Nevertheless "mathematically impossible" does not mean "self-contradictory or leading to self-contradictions". We must pay attention not only to the way mathematicians claim the impossibility but also the consequences, in particular the fact that they dismiss the impossibility from consideration in mathematical theory. Therefore "*contradictoriness can be, mathematically speaking, a criterion of impossibility*" (p. 30) while "*the force of calling the number x an 'impossibility' is to dismiss it from consideration*" (p. 30).

Argumentation and mathematical proof

Applying the model to a mathematical proof, we see that a common – but not the only (see Winicki-Landman, 2007) - method to prove an impossibility is proof by contradiction. In this type of proof, non-Q is the statement to be proved, P is the mathematical theory within which the proof makes sense (Euclidean geometry, number theory, etc.) and R is a contradiction or a proposition that contradicts a statement included in P. Accepting R causes a penalties S, that is a self-contradictory mathematical theory P. A logical statement makes explicit that the contradiction is a criterion for impossibility, stating that if Q implies a contradiction, non-Q is valid and then Q is not valid.

In general, in students' argumentation but also in those produced by experts with goals different from that of validate a statement in a theory, some parts of the pattern could have a different nature. P could comprehend visual considerations, beliefs, misunderstandings, etc, and also R and S could take into account extra-mathematical factors. As Toulmin stated (p. 28): "*the offence involved (R) and the penalties risked (S) also vary from case to case*". Sometimes, R could stand for a figure with strange lines or angles that bad represents a geometrical concept (as it happens in proof by contradiction), an uncommon proposition, a situation not expected because of the didactical contract, etc. and S could be the contradictoriness of a theory, the loss of meaning of words or figures, a penalties for having broken the didactical contract, and so on. Finally, students could not share that the contradiction is a criterion for impossibility (see Antonini, 2008) even if they are not aware of it. In these cases, they could try to eliminate the contradiction, interpreting it in different ways (Antonini, 2008, Mariotti & Antonini, 2009). In the following, I propose two examples in which we will see these aspects in students' argumentations.

TWO EXAMPLES

The following examples are drawn from a wide-ranging research study concerning proof (see Antonini & Mariotti, 2008) and an unpublished research on production of examples. Collection of data of these studies was carried out through various means, interviews, questionnaires, recording and transcripts of classroom activities, and involved students at the high school (12^{th} and 13^{th} grade) and at the University level (Scientific Faculties such as Mathematics, Physic, and Biology).

The impossible triangle

Elenia and Francesca (first year of the Biology Faculty) are asked to investigate the following problem: *What can you say about the angle formed by two angle-bisectors in a triangle?* After a short exploration, the students consider the possibility that the angle is right and deduce that "if the angle between the angle bisectors is right then $2\alpha+2\beta=180$". I propose a transcript of interview that starts immediately after this point and in which only Elenia is speaking:

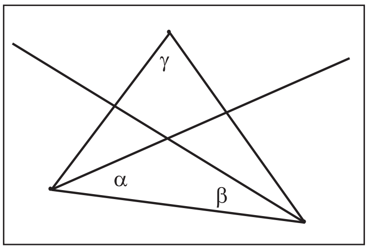

Figure 1

46 E: … there is something wrong.

47 I: Where?

48 E: In 180. […]. Because, it [180] is the sum of all the three interior angles, isn't it? […]

55 I: And then?

56 E: And then there is something wrong! They should be $2\alpha+2\beta+\gamma=180$ […]

60 E: …and then it would become $\gamma=0$… […]. But equal to 0 means that it isn't a triangle! If not, it would be so [she joins her hands]. Can I arrange the lines in this way? No... […]

85 E: And then essentially there is no triangle any more.

86 I: And now?

87 E: …that it cannot be 90 [degrees] […] because, in fact, if $\gamma=0$ it means that… it is as if the triangle essentially closed on itself and then it is not even a triangle any more, it is exactly a line, that is absurd.

The proposition '$2\alpha+2\beta=180°$ ' is correctly deduced and contrasts with a theorem of the theory, then it produces a contradiction. This would be sufficient to validate the impossibility that the angle between the angle bisectors is right, because leading to a contradiction is a criterion for mathematical impossibility. Nevertheless, Elenia seems to be surprised and puzzled (46-56) and, in the first part of the excerpt, she looks at the proposition as a mistake ("*there is something wrong*"), an impossibility that, as pointed out in the theoretical framework, is dismissed from consideration. In fact, Elenia initially refutes to look for some consequences of this proposition. Further investigation leads her to conclude that "$\gamma=0$" and to generate a new figure, a line. At this point, the proposition becomes possible, and then it is entitled to consideration. It appears that two superimposed lines is a possible geometrical situation that one can take into account while a triangle in which the sum of two angles is equal to 180° is an impossible situation that one is inclined to dismiss.

The difference between impossibility and contradictoriness appears here. Initially, the proposition '$2\alpha+2\beta=180°$ ' is seen as an impossibility (that cannot be used in argumentation) and not a contradiction that can support an impossibility. Only when the situation is considered possible – because she manages to give it a geometrical meaning - Elenia can continue her exploration (for an analysis of this interview related to some cognitive aspects of geometry, see Mariotti & Antonini, 2009).

I observe that the student does not make explicit every part of the model. In particular we do not find the penalties for accepting that a triangle is a line. However, we can hypothesize that, for Elenia, the penalties are the change or the loss of meaning of the geometrical categories, in particular of the word "triangle".

Finally, according to the Toulmin's pattern, the argumentation can be modelled in the following way: *the geometrical figures being as they are, the angle between the angle bisectors of a triangle cannot be right: otherwise there would be not a triangle or the triangle would be a line (and this would provoke a change or the loss of meaning of the word "triangle")*.

The impossible function

Federica (fifth year of the Mathematics Faculty) is asked to work on this problem: *give an example, if possible, of a twice differentiable function* $f:[a,b]\rightarrow R$*, such that f is zero in three different points and its second derivate is positive in the domain*.

Federica: We should manage to join two functions […] in a smooth way so that the result is differentiable. [...] I give you an example [see fig. 2], this function is zero in at least three points but it doesn't work because there is a point where it is not differentiable.

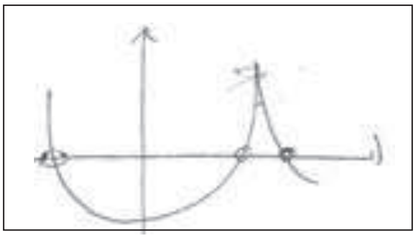

Figure 2

Federica: The problem is that, I ask myself if in order to have the derivability in points like this, I have necessarily to consider a piece of function that is concave; or, if not concave, constant, that is not good because the second derivate is zero.

I omit a part of the interview in which Federica tries to construct the function by defining the analytic expressions in two adjacent segments and in the point that separates the segments. After this work she realizes that the problem is again in joining the expressions so that the requirements are fulfilled and she produces a conjecture and an argumentation:

Federica: I suspect that it is absurd. Because with functions like that I wrote, when I define [the value of the function] in one point I lose the second derivate everywhere positive. However, if I define it by piecewise it is not easy to join them [the pieces] so that it [the function] is twice differentiable. […] If the function were … […] Let's see what happens in n [*she is assuming that the function is composed by two convex functions joined in a point named n*] […] I would like to show that the function in n either isn't continue or isn't differentiable, in order to arrive at an absurdity.

According to the Toulmin's pattern, the Federica's argumentation can be modelled in this way: *the concepts of calculus and the graphs of functions being as they are, you cannot give a function with all the requested properties, in particular you cannot construct it joining two convex functions: otherwise there would be a piece of function that is concave, constant, not continue or not differentiable, transgressing the requirements of the task*.

The criterion of impossibility seems to be linked to the actual constructability: the function cannot exist because Federica manages neither to draw the graphs nor to write an analytic expressions.

We can observe here a continuity (in the sense of Garuti *et al.*, 1996 and Pedemonte, 2007) between the structure of argumentation and that of the projected proof. In the last part of the excerpt, Federica plans to construct a proof by contradiction but some limitations appear. First of all, she does not assume only that the function exists, but, in continuity with the precedent stage, she assumes that the function is composed by two convex functions joined in the point n. In addition, she plans not to look for any contradiction but a proposition regarding the joining point n that is the negation of some required properties (concavity, non differentiability, etc.). The Toulmin's model allows us to emphasize that the structure of the planned proof is the same of that of the argumentation. The main difference between them is the part P of the model, that, in the stage of proof construction (here omitted for the limitation of pages), regards in particular the mathematical theory and not visual aspects. Moreover, the other parts remain the same: Federica plans to show that in the point n the function is not differentiable or not continue, and this would transgress the requirements of the task. Because Federica looks for a proposition that transgresses the task and not any contradiction, we cannot say that she accepts contradiction as a criterion for impossibility.

CONCLUSIONS

The Toulmin's model proposed in this paper seems suitable for analysing students' argumentation supporting impossibilities. Of course, it can happen that some parts of the model remain implicit and, in any case, further researches are needed to investigate if other different structures could be identified in these types of argumentations and to set up different or more general models.

I observe that the model proposed here is a refinement of the scheme of indirect proof described by Freudenthal (1973, p. 629) ('*... if it were not so, it would happen that...*') that, applied to impossibility assumes the following form: '*if it were so, it would happen that*'. Differently, the Toulmin's model does not appear similar to that set up by Antonini & Mariotti (2008) to analyse indirect proofs. Studies are needed to investigate differences, analogies and possible integrations between these two models.

The Toulmin's model has allowed identifying some important aspects both from epistemological and didactical points of view. The structure of argumentations, the criteria of impossibility and the notion of impossibility and of possibility can offer a good base to explain difficulties students have with this type of argumentations. It follows also the importance for students to deal with activities involving concepts like impossibility, contradictoriness, false, absurd, etc. and to be guided in distinguishing these notions.

The elements of the model can offer a contribution to the analysis of differences and analogies between proof and argumentation in situation involving impossibilities. In particular it could allow identifying aspects that are specifically cultural in mathematical proof of impossibility and then that require a specific education, and aspects of continuity that, according to Garuti *et al.* (1996), can be used in an approach to proof based on activities involving argumentations.

Further studies are needed to investigate the criteria of impossibility, that depend on the field but also on the subject, who could not share the mathematical criteria (see Antonini, 2008), and on the function of the argumentation. One of the goal of mathematical education should be to make students aware of different criteria of impossibility and to introduce them to criteria that have a cultural nature and that are shared by scientific community of mathematicians.

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