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RESEARCH FORUM 2

ARGUMENTATION AND PROOF: A CONTRIBUTION TO THEORETICAL PERSPECTIVES AND THEIR CLASSROOM IMPLEMENTATION

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ARGUMENTATION AND PROOF: A CONTRIBUTION TO THEORETICAL PERSPECTIVES AND THEIR CLASSROOM IMPLEMENTATION

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The complexity of proving and the relationship between argumentation and proof are subjects of major concern in Mathematics Education. In this Research Forum we will propose the integration of Toulmin's model of argumentation with Habermas' elaboration of rational behavior, both adapted to proof and proving. After a short presentation of Toulmin's and Habermas' tools at work, we will provide some theoretical reasons for their integration. Then we will show how this construct allows us to frame the planning, management and analysis of some classroom activities aimed at students' approach to relevant aspects of proving and proof. Finally, we will suggest some further developments. A reaction by Carmen Samper and her colleagues (related to their own research work) will introduce the discussion.

1. INTRODUCTION

Since the late nineteen eighties, mathematical proof and proving have been one of the main subjects of research in mathematics education. Different strands of research have developed since that time. In particular, several mathematics educators (starting with Gila Hanna, Nicolas Balacheff, Raymond Duval and others: see Balacheff, 1987; Hanna, 1989; Duval, 1991) have considered, within an educational perspective, the relationships, tensions, and potential oppositions between: formal proof, and semantic or informal argument; proof as a cultural product subject to logical and communicative (textual) constraints, and proving as the process aimed at that product; mathematical proof, and ordinary argumentation. In spite of the apparent, broad differences, all these relationships deal with a crucial dichotomy: on the one hand, we have to consider mathematics output as fitting a set of rules, constraints, logical and textual models; on the other hand, there is the creative and constructive side of mathematicians' (and students') activity when they are engaged in understanding and validating mathematical statements, using cultural processes.

The theoretical background consists of two main constructs (Toulmin's model for argumentation, see Toulmin, 1974; and Habermas' construct of "rational behavior"- see Habermas, 2003 - as adapted in Boero & Morselli, 2009); other constructs will be discussed in specific instances (Peirce's abduction; cognitive unity and structural

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continuity - see Pedemonte, 2007, 2008; phases of proving - see Arzarello, 2007; Douek, 2009). Our research provides evidence that:

a. Toulmin's model of argumentation can be used to analyze and compare the organization of arguments between the exploratory, creative phases and the deductive chaining of the resulting arguments (see Section 2).
b. Habermas' construct of rational behavior can be used to analyze the components of the proving process (and the tensions between these): the conscious assumption of the epistemic constraints (related to mathematical deduction) inherent in the desired product; the teleological component (i.e. the conscious choice of the tools to achieve the aim of the activity); and the communicative component (i.e. the conscious choice of tools in order to conform the product to the textual constraints of proof, or to be understood by others) (see Section 3).

The Research Forum will focus on the need to integrate the two models and on a proposal for integrating them. Indeed, if considered separately, Toulmin's model is suitable to analyze the organization of arguments, not the subject's intentions nor the tensions between the different "components" of the proving process; conversely, Habermas' construct is suitable to take the subject's intentions and consciousness into account, but it does not offer the possibility of modelling and comparing different kinds of productions (in particular, the composition of arguments in the exploratory phases and in the final proof) and eliciting possible continuities and discontinuities (and related obstacles) between them.

The main goals of the Research Forum are:

1. to present the integrated theoretical frame (Section 4);
2. to show how such a frame can be used for the design, management and analysis of teaching and learning activities regarding proof (Section 5).

In a Vygotskian perspective (Boero, 2006), the contribution made by this Research Forum concerns some aspects of the students' approach to scientific knowledge in the case of theorems: in particular, awareness and intentional use of knowledge are promoted through the direct mediation of the teacher in the interplay between mathematical discussion and the story narration of proof construction (see Section 5). Some research and educational developments will be discussed in Section 6.

2. TOULMIN'S MODEL

In the educational literature, Toulmin’s model has been already used for analyzing and documenting both how learning progresses in a classroom (Krummehuer, 1995) and how to create a context for arguing in the class (Wood, 1999). The use of this model in our previous research was critical for comparing students’ argumentations with their proofs, from a structural and cognitive point of view (Pedemonte 2005, 2007, 2008). This comparison is based on the hypothesis that proof is a particular type of argumentation in mathematics (Pedemonte, 2007). According to linguistic theories (Plantin, 1990; Toulmin, 1974) proof is a set of rational justifications expressed as inferences. These inferences are analyzed and compared using Toulmin's model.

2.1.Toulmin's model: a methodological tool to analyze argumentation and proof

In Toulmin's model an argument consists of three elements (Toulmin, 1974):

C (*claim*): the statement of the speaker
D (*data*): data justifying claim C,
W (*warrant*): the inference rule, which allows data to be connected to the claim.

In any argument, the first step is expressed by a viewpoint (an assertion, an opinion). In Toulmin's terminology the standpoint is called the claim. The second step consists of providing data to support the claim. The warrant provides the justification for using the data as a support for the claim. The warrant, which can be expressed as a principle a rule, acts as a bridge between the data and the claim. Three other elements that describe an argument can be taken into account: B (*backing*) the support of the rule; Q (*qualifier*) the strength of the argument; Re (*rebuttal*) the exception to the rule. The force of the warrant would be weakened if there were exceptions to the rule: in that case conditions of exceptions or rebuttal should be inserted. The claim must then be weakened by means of a qualifier. Backing is required if the authority of the warrant is not accepted straight away. Overall, Toulmin's model of argumentation contains six related elements as shown in Figure 1.

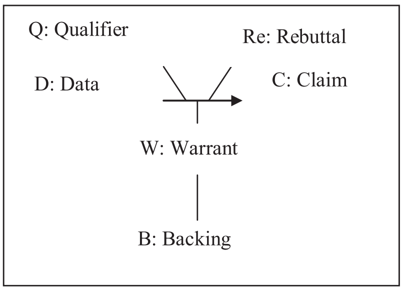

Figure 1: Toulmin's model of argumentation

To understand how Toulmin's model is applied, consider the following argument given by a student to answer the question[1]:

> "What can you say about $-a^2$ if a is an integer number different from 0? Is it a positive or a negative number?"
>
> "$-a^2$ is a negative number (*claim*) because the square of each number is a positive number, but with minus it becomes a negative number (*warrant*)… unless the square is made for the whole number and the minus… in this case $-a^2$ is a positive number (*rebuttal*). No… this is impossible because $-a^2$ is different than the square of $(-a)^2$"

The argument can be illustrated as follows:

Q: probably *unless* Re: $-a^2>0$ because $-a^2 = (-a)^2>0$

D: $-a^2$, $a\in Z$, $a\neq 0$ $\longrightarrow$ C: $-a^2<0$

since W: $n^2\geq 0$ $\forall n\in Z$; if $n>0$ then $-n<0$

on account of B: Algebraic rules and in particular $-a^2 \neq (-a)^2$

Figure 2: Toulmin's model example

2.2. Why is it important to compare argumentation and proof?

The relationship between conjecturing and proving has been analyzed in mathematics education from different points of view and with different educational aims. On the

[1] This answer was given in a test designed to evaluate algebraic competencies in a class of 13-14 year-olds.

one hand, some research suggests major differences between argumentation and proof both from a social and epistemological point of view (Balacheff, 1987) and from a cognitive and logical point of view (Duval, 1991). On the other hand, some Italian studies highlight the continuity that may exist between argumentation as a process of producing a conjecture and constructing its proof (Boero, Garuti, Mariotti, 1996). This continuity is called *cognitive unity*. During the problem-solving process, argumentation is usually required in order to produce a conjecture. The hypothesis of cognitive unity is that, in some cases, this argumentation can be exploited by the student in the construction of a proof by organizing some of the previously produced arguments into a logical chain. Research on cognitive unity (Boero & al., 1996; Garuti, Boero, Lemut, Mariotti, 1996) shows that proof is more "accessible" to students if an argumentation activity leads to the construction of a conjecture, which is then proved by building on the argumentation. Following these research studies, Pedemonte (2007) has shown that the analysis of cognitive unity does not cover all aspects of the relationship between argumentation and proof. In particular, it seems very important for a cognitive analysis of argumentation and proof to consider two points of view:

1. the *referential system*, made up of the representation system (the language, heuristics, and drawings) and the knowledge system (conceptions and theorems) of argumentation and proof (Pedemonte, 2005). The analysis of *cognitive unity* takes into account the referential system.
2. the *structure* intended to allow logical cognitive connection between statements (deduction, abduction, and induction structures) (Pedemonte, 2007).

There is continuity in the referential system between argumentation and proof if some expressions, drawings, or theorems used in the proof have been used in the argumentation supporting the conjecture. There is structural continuity between argumentation and proof when inferences in argumentation and proof are connected through the same structure (abduction, induction, or deduction). For example, there is structural continuity between argumentation and proof if some abductive steps used in the argumentation are also present in the proof. It is important to observe that if continuity in the referential system between argumentation and proof supports the construction of proof, this is not generally the case for the structural continuity. As a matter of fact, to produce a deductive proof, a structural distance is in many cases necessary because the structure of argumentation is usually not deductive. Sometimes students are unable to construct a proof because there is spontaneous continuity between argumentation and proof (e.g. from abductive argumentation to a sort of "abductive" proof: see the example in section 2.3). Toulmin's model is an important and effective tool to analyze argumentation and proof because the two kinds of analysis – the structural analysis and the referential system analysis – can be performed using it.

Toulmin's model as a way to analyze and compare the referential system and the structure between argumentation and proof

A warrant in a proof is an axiom, definition, or theorem. Backing is the theoretical system which justifies the warrant. In the argumentation supporting the conjecture, these elements do not necessarily belong to a theoretical system. If the warrant is a mathematical rule there will probably be continuity between argumentation and proof because this rule can be replaced in the proof with a theorem (Pedemonte 2005). In contrast, if the rule is not correct, it cannot be replaced by a theorem in the proof. In this case, three possibilities can be identified: the proof is not constructed by the student (the *cognitive unity is broken*), an incorrect "proof" is constructed and is based on the incorrect rule used in the argumentation (*cognitive unity*); the incorrect argumentation is abandoned, another argumentation is constructed and turned into a correct proof (the *cognitive unity is broken but a successful argumentation is constructed*).

In Toulmin's model a step appears as a deductive step: data and warrants lead to the claim. However, other argumentative structures can be represented by this model (abduction, induction, etc.). For example, an *abductive* step can be expressed as follows (the question mark means that data are to be sought in order to apply the inference rule justifying the claim).

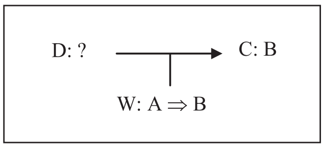

Figure 3: Abduction in Toulmin's model

2.3. Example

In this section, we describe the resolution process (taken from a set of data collected from a teaching experiment carried out in some traditional 12th and 13th grade classes in France and Italy) for an open-ended geometry problem.

> Problem: ABC is a triangle. Three exterior squares are constructed along the triangle's sides. The free points of the squares are connected, defining three more triangles. Compare the areas of these triangles with the area of triangle ABC.

The area of each of these triangles is equal to the area of triangle ABC. The solution to this problem is not obvious to students. In order to find a successful strategy, an abductive argumentation is usually constructed, as we show in this example. Note that, in the proof, abductive steps are present because students were not able to transform argumentation in a deductive proof. This example is representative of the vast majority of analyzed resolution processes. The analysis start at claim C_6 (the conjecture); at this point the students affirm that the area of triangle ABC and the area of triangle ECL (see figure) are equal. So far, they have calculated these areas.

103.C : The areas are always equal … with the calculator the areas are equa 104. N: Now we have to see why!	The figure as represented by the student using Cabri Geometry  C_6: The triangles' areas are equal

Students have to justify the claim C_6, which is constructed as a “fact”. Two abductive steps are developed in the argumentation.

The first is needed to justify the equality of the triangles' areas. The students look for a relationship between bases and heights that make a constant area (step 6).

The second step is needed to justify that this relationship always holds. The triangle bases are congruent, so students tend to justify the congruence of the heights (step 7).

.

105. C : We need to find how the base and th height change …if there is a relationship tha makes the area constant… The area is constant.. but I don't understand…so we have to find base by height congruent to base by height of the othe triangle 106. N: If we take the constant bases and w change the heights...	D_6: ? $\longrightarrow$ C_6 W: Formula of the area If the relationship between bases and heights has to b constant, the heights have to be congruent because th bases are congruent. D_7: congruent bases ? (congruent heights) $\longrightarrow$ C_7= D_6: CB*AN =CL*ED W: transitivity of congruence

The comparison between heights leads explicitly to a comparison between the two small triangles ANC and EDC constructed on the heights. Students try to establish their congruence. If the two triangles are congruent, their heights are congruent. This step is an abductive step (step 8). As in the previous example, students look for data to apply the congruence criterion and justify the congruence between the two small triangles ANC and EDC (step 9). Again the argumentation structure is abduction.

115. C: But why are the heights congruent? 116. N: We have…. we see that this side is congruent to this side of triangle ABC … 117.C: Then the small triangle is congruent to the other small triangle … 118. N: … yes it's true, two sides are congruent 119.C: So there is a 90° angle 120. N: We need another side or another angle…	, ?D_8: ANC=EDC $\longrightarrow$ C_8: D_7 W: Inheritance of congruence ?D_9 $\longrightarrow$ C_9: D_8 W: SAA congruence criterion

Once students find data to justify the congruence between the two small triangles ANC and EDC, they can construct the proof.

Proof: The protocol appears to be an example of cognitive unity. But there is no structural change between argumentation and proof. The transcript of this proof describes the students' abductive reasoning. The students are unable to completely cover the distance between argumentation and deductive proof. Argument 7 is still an abductive step (congruency between heights has to be proved). This is the reason why in this case we can observe a structural continuity between argumentation and proof.

<table>
<tr>
<td>Students consider the triangles ABC and ELC
We know that this base is congruent to th
base of the triangle. Now we have to prove tha
the heights are congruent. We have verifie
this fact by means of the congruence criterio
proved on the sheet with the drawing.
On the sheet with the drawing:
Triangle ANC = Triangle EDC
EC=AC
EDC=ANC=90°
ACN=ECD because ACE=90°, DCN=90° an
if the angle DCA is removed from the tw
other angles we have the same angle.</td>
<td>D_7: congruent bases
? (congruent heights) → C_7: The areas of triangles are equal
W: area formula

D_8: ANC=EDC → C_8: congruent heights
W: inheritance of congruence

D_9: EC=AC
EDC=ANC=90°
ACN=ECD → C_9 : D_8
W: SAA congruence criterion</td>
</tr>
</table>

2.4. Some comments and further considerations

Some experiments in geometry classes (Pedemonte, 2007) highlighted that this "spontaneous" structural continuity between abductive argumentation and "abductive proof" can be an obstacle for students in the construction of a deductive proof. However, this continuity between abductive argumentation and proof seems to be absent when students produce algebraic proofs. As a matter of fact, unlike in the geometry example, the structural distance between argumentation and proof (from an abductive argumentation to a deductive proof) is not one of the common difficulties met by students in solving problems involving properties of numbers. Since algebraic proof is characterized by a strong deductive structure, abductive steps in the argumentation activity can be useful in linking the meaning of the letters used in the algebraic proof with numbers used in the argumentation. In fact, the strength of the deductive structure in algebraic proof prevents, at least partially, the occurrence of structural continuity between argumentation and proof (Pedemonte, 2008). Moreover, in this case analysis performed through Toulmin's model has shown that, unlike in the geometrical case, abductive steps in the argumentation can be useful for the construction of proof if they favor continuity in the "content" of argumentation and proof (sometimes abductive steps assume an important role in the argumentation because, through them, students' reasoning maintains the connection between construction of a conjecture and proof).

3. PROVING AS A RATIONAL BEHAVIOUR: HABERMAS' MODEL

Balacheff (1982) points out that the teaching of proofs and theorems should have the double aim of making students understand what a proof is and making them learn to produce proofs. Accordingly, we think that proof should be dealt with in mathematics education by considering both the object aspect (a product that must meet the epistemic and communicative requirements established in today mathematics - or in school mathematics) and the process aspect (a special case of problem solving: a process intentionally aimed at a proof as product) of proof. We have tried (Boero, 2006; Morselli, 2007) to match these considerations with Habermas' elaboration about rationality in discursive practices; we will present here a unified synthesis of our previous work. Habermas (2003, ch.2) distinguishes three interrelated components of rational behaviour: the epistemic component (inherent in the control of the propositions and their chaining), the teleological component (inherent in the conscious choice of tools to achieve the goal of the activity) and the communicative component (inherent in the conscious choice of suitable means of communication within a given community). With an eye to Habermas' elaboration, in the discursive practice of proving we can identify: A) an **epistemic aspect,** consisting in the conscious validation of statements according to shared premises and legitimate ways of reasoning (cf. the definition of "theorem" by Mariotti & al. (1997) as the system consisting of a statement, a proof, derived according to shared inference rules from axioms and other theorems, and a reference theory); B) a **teleological aspect**, inherent in the problem-solving character of proving, and the conscious choices to be made in order to obtain the desired product; C) a **communicative aspect**, consisting in the conscious adhering to rules that ensure both the possibility of communicating steps of reasoning and the conformity of the products (proofs) to standards in a given mathematical culture.

Our point is that considering proof and proving according to Habermas' construct may provide the researcher with a comprehensive frame within which to situate a lot of research work performed in the last two decades (see below for some examples), to analyze students' difficulties concerning theorems and proofs (see the two examples in the next subsection), and to discuss some related issues and possible implications for the teaching of theorems and proof (see Boero, 2006; Morselli & Boero, 2008; 2009).

Regarding the epistemic aspect, i.e. in the analysis of proofs and theorems as objects, mathematics education literature offers some historical analyses (e.g. Arsac, 1988) and surveys of epistemological perspectives (e.g. Arzarello, 2007). These help us understand how theorems and proofs originated and how they were perceived in different historical periods and how, even today there is no fully shared agreement about what constitutes a "mathematical proof" (cf. Habermas' comment about the historically and socially situated character of epistemic rationality). Also relevant to epistemic rationality is Duval's (2001) focus on the distance between mathematical proof and ordinary argumentation (by referring proof to the model of formal derivation). He also considers how to make students aware of that distance and able to manage the construction and control of a deductive chain. Harel (2008) uses the DNR theoretical framework to frame the classification of students' proof schemes (they concern proof as a final product). We note that, in terms of Habermas' components of rationality, Harel's ritual and non-referential symbolic proof schemes may be attributed to the dominance of the communicative aspect in an educational context where students' intentions are driven by the necessity of conforming to the supposed requests of the teacher, with lacks inherent in the epistemic component (cf. Harel's N, "intellectual Necessity").

Concerning the proving process, some analyses of its relationships with arguing and conjecturing suggest possible ways to enable students to manage the teleological rationality. In particular, Boero, Douek & Ferrari (2008) focus on the existence of common features between arguing and proving processes, and they present some activities (designed for grade 1 on), based on those commonalities, that may prepare students to develop effective proving processes.

3.1 Two examples of analysis within the frame of rationality

Morselli (2007) investigated the conjecturing and proving processes carried out by different groups of university students (7 first-year and 11 third-year mathematics students, or 29 third-year students preparing to become primary school teachers). The students were given the following problem: *What can you tell about the divisors of two consecutive numbers? Motivate your answer in general.* Different proofs can be carried out at different mathematical levels (by exploiting properties of the remainder, examining multiples, or using algebraic tools). The students worked on the problem individually, writing down their process of solution. Afterwards, students were asked to reconstruct their process and comment on it; these interviews were audio taped. Morselli (2007) provided several examples of individual solutions and related interviews, in particular showing how students' failures or mistakes were due to lacks in some aspects of rationality and/or the dominance of one aspect over the others. We will consider here only portions of two very similar examples, concerning students that are preparing to teach in primary school, in order to show how Habermas' construct works as a tool for in-depth analyses.

Example 1: Monica

Monica considers two couples of numbers: 14, 15 and 24, 25. By listing the divisors, she discovers that "Two consecutive numbers are odd and even, hence only the even number will be divisible by 2". Afterwards, she lists the divisors of 6 and 7 and writes: "Even numbers may have both odd and even divisors". After a check on 19 and 20, she writes the discovered property, followed by its proof:

> Property: two consecutive numbers have only one common divisor, the number 1. In order to prove it, I can start by saying that two consecutive numbers certainly cannot have common divisors that are even, since odd numbers cannot be divided by an even number. They also cannot have common divisors different from *1*, because between the

> two numbers there is only one unit; if a number is divisible by *3*, the next number that is divisible by *3* will be greater by *3* units, and not by only one unit. Since *3* is the first odd number after *1*, there are no other numbers that can work as divisors of two consecutive numbers.

Monica carries out reasoning that is intentionally aimed (teleological aspect): first, at the production of a good conjecture; then, at its proof. Proof steps are justified one by one (epistemic aspect) and communicated with appropriate technical expressions (communicative aspect). The only lack in terms of rationality concerns the short-cut in the last part of the proof: Monica realizes that something similar to what happens with 3 (the next multiple is "greater by three units") will happen *a fortiori* with the other odd numbers that are bigger than 3 ("Since 3 is the first odd number after 1"), but she does not make it explicit. Her awareness (cf. epistemic rationality) is not communicated in the explicit mathematical form (lack of communicative rationality). Monica's later comments on her text confirm the analysis:

> Monica: (...) and then I thought that 3 was the first odd number after 1 and so if 3 does not enter there, also the bigger ones do not enter there [from the previous text, "there" means: between two consecutive numbers on the number line].
>
> Observer: To make more general what you said with 3, what would you write now?
>
> Monica: ehm... I have tried to go beyond the specific case of 3, but I do not know if I have succeeded.

Example 2: Caterina

> Starting from the fact that two consecutive numbers are always one odd and one even, we may conclude that the two numbers cannot be both divided by an even number. Afterwards, we focus on odd divisors; we start from 1, and we know that all numbers may be divided by 1; the second one is 3. We have two consecutive numbers, then the difference between them is 1, then they will not be multiples of 3, since it will be impossible to divide both of them by a number bigger than 1.

Caterina is able to justify all the explicit steps of her reasoning (epistemic aspect), she develops goal-oriented reasoning (teleological aspect) and illustrates her process with appropriate technical expressions (communicative aspect). In spite of good intuition, there is a lack in her reasoning: she does not consider divisors greater than 3 (unlike Monica). Later, after having seen also the production of her colleagues, Caterina comments:

> My reasoning is not mistaken: indeed, I reach the conclusion giving a general explanation, saying that, since there is no more than one unit between the two numbers, the only common divisor is 1. Nevertheless, I cannot create a mathematical rule. Observing the other solutions, I think that the correct rule is the following: along the number line we note that a multiple of 2 occurs every two numbers, a multiple of 3 occurs every three numbers, and hence a multiple of N occurs every N numbers. Then, two consecutive numbers have only 1 as common divisor.

From the objective point of view of epistemic rationality, Caterina's argument was not complete, though her comment reveals that she is not aware of it. From her subjective point of view, Caterina is convinced to have found a cogent reason for the validity of the conjecture ("not mistaken reasoning", "general explanation"), thus to have achieved her goal (teleological rationality). Some colleagues' solutions induced her to reflect on the lack of a "mathematical rule." However, from her comment it seems that this lack is not considered by her as a lack in the reasoning but as a lack in the mathematical communication.

3.2. Habermas' construct and the use of algebraic language in proving

In this Subsection we briefly consider the use of algebraic language in proving and show how Habermas' construct can be further specialized to frame this. Algebraic language can play the role of a tool for proving through modelling (see Norman, 1993, and Dapueto & Parenti, 1999) within mathematics (e.g. when proving theorems of elementary number theory).

Our interest for considering the use of algebraic language in the perspective of Habermas' construct depends on the fact that our previous research (Boero, 2006; Morselli, 2007) suggests that some of the students' main difficulties in conjecturing and proving depend on specific aspects (already pointed out in literature) of the use of algebraic language. Difficulties with these aspects make conjecturing and proving complex and demanding activities for students. In particular, we refer to: an epistemic component, inherent in the need to check the validity of algebraic formalizations and transformations and to correctly and purposefully interpret algebraic expressions in a given context of use; a teleological component, inherent in the goal-oriented character of the choice of formalisms and of the direction of transformations; a communicational component, inherent in the restrictions that come from the need to follow taught communication rules, which may contradict private rules of use or interfere with them. In Morselli & Boero (2009) we have tried to show how framing the use of algebraic language with the perspective of Habermas' theory of rationality: first, provides the researcher with an effective tool to describe and interpret some of the main difficulties met by students when using algebraic language in proving; second, provides the teacher with some useful indications for the teaching of algebraic language; third, suggests new research developments.

4. THE INTEGRATION OF HABERMAS' AND TOULMIN'S MODELS

As we have seen in the previous sections, we think that Toulmin's model for argumentation and Habermas' model for rational behavior, suitably adapted to the specificity of "mathematical proving", are valuable analytical tools to deal with some relevant aspects of the complexity of the conjecturing and proving activity. These models can be useful to analyze students' performances and difficulties in developing proving skills. In particular, those analytical tools allow one to evaluate the distance between the actual behavior of students and the behavior that teachers would like to promote, such as the case of the relationships between argumentation and proof (in the delicate transition from the argumentative search for reasons of validity of a statement, to their organization and chaining in a mathematical proof - see Section 2) and the case of the rational behavior inherent in proving, with its epistemic, teleological and communicative components (see Section 3). Educational implications include pointing out some negative features of the traditional school approach to mathematical proof: for instance, Habermas' model reveals that teachers' pressure on communicative and epistemic rationality (according to teacher's culture) usually prevails over the need for teleological rationality (see Morselli & Boero, 2009). The models also suggest some guidelines for preparing teachers and planning teaching (see Boero 2006, Morselli & Boero, 2009) with sensitivity to students' conceptions. However, neither model on its own seems to offer a comprehensive frame for classroom implementation of teaching of mathematical proof, especially concerning practices that should allow students to develop consciousness about the requirements of proving in mathematics and the ways of managing proving. The contribution that we would like to bring in this Section is a proposal to integrate Toulmin's and Habermas' analytical tools, in order to get a frame suitable for (1) combining them in the analysis of students' behavior, and (2) inspiring and planning innovative classroom practices aimed at developing students' awareness of the nature of proving. By their nature, the integration of the two models can work as an integration of two different levels of detail and focus of the analysis: when looking at the (oral and written) texts produced by students through Toulmin's lens, the unit of analysis is the fundamental step of argumentation which connects data to claim based on a warrant (with possible expansions in the context of the argumentation, and links of different kinds - deductive chaining, opposition, etc. - with other steps). When looking at the behavior with Habermas' lens, the unit of analysis is related to the motive of one specific phase of the activity (e.g. the production of the final text of the proof, or the organization of some arguments into a reliable chain), or to a specific strategy of proving (e.g. that related to the use of algebraic language). Here we can see how the components of Habermas' rationality are useful to focus on the legitimacy of reasoning steps, on the specificity of the intention, and on the communicational constraints.

The two models complement each other in the following sense. In the case of the expert, the process underlying the discursive behavior of proving develops under (more or less conscious) constraints of epistemic validity, efficiency related to the goal to achieve, and communication according to shared rules. Those constraints (particularly, but not only, the first one!) result in two levels of argumentation: the level (we could say, a meta-level) inherent in the awareness of the constraints on the three components of rational behavior in proving; and the level inherent in the specific nature of the three components. The protagonists of the classroom scene (the teacher and the students) develop different levels of argumentation. Initially, the teacher guides students' activity, applying his/her meta-level of argumentation. Thus, students become aware of these components and try to conform to them when working out their argumentation, until they can justify their steps of reasoning from the perspective of these components. Meta-level is not a goal for students, it is a teaching means.

The proposed integration of Toulmin's and Habermas' models allows us to look at the enculturation in the "culture of theorems" managed by the teacher as a process in which the teacher (through suitable tasks like the "story narration" of proof construction that will be presented in the next Section) offers to students the opportunity to enrich the level of argumentation used in the justification of a statement (on shared epistemic bases), with the meta-level concerning the awareness of the epistemic, teleological and communicative requirements of proving. Warrants spontaneously referred to by students mainly concern the epistemic aspect of proving and can be visual evidence (in the case of geometry), properties obtained through algebraic transformations, and properties for which no doubt exists. At the meta-level that must be promoted by the teacher, warrants concern the reliability and the nature of the "epistemic warrants" ("the conclusion is not yet legitimate *because we can use only agreed properties and already proved statements*"), the efficiency of the strategies and the choices adopted to achieve the proof ("we use algebraic language *because it allows us to express ideas precisely*..."), the rules of communication in mathematics ("this text is not yet satisfactory *because all the steps of reasoning must be made explicit with the correct words*"). We can observe that in the case of epistemic rationality, backing at the meta-level refers to the reference theory and the related rules of inference.

According to the "integrated" model of Toulmin and Habermas, the organization of reasoning in terms of warrants must be consciously assumed by the student as one of the main aims of the activity. The mathematical discussion orchestrated by the teacher (Bartolini Bussi, 1996) appears as an important organization of the collective classroom activities aimed at promoting the argumentation of students at the meta-level with the goal of developing rational behavior in proving. In interplay with classroom mathematical discussion, the story-narration of proof construction will aim to produce argumentative texts in which the search for warrants and their presentation at the meta-level become the main goals of students' activity.

In order to frame the educational choices and the teaching experiments that will be presented in the next Section, we need to focus on phases of proving activity and some specific features of argumentation in proving.

Three modes of reasoning

Inspired by Lolli's analysis of proof production (see Arzarello, 2007), we consider proving as a cognitive, culturally situated activity engaging three modes of reasoning:

- Mode 1: exploration and production of reasons for validity of the statement;
- Mode 2: organisation of reasoning into a cogent argumentation;
- Mode 3: production of a deductive text according to specific cultural constraints concerning the nature of propositions and their chaining.

For educational aims, these reasoning modes can be considered as successive phases of a proof construction. But in the process of proving, they do not appear separately, rather they are deeply connected, and their succession may vary and loop. We will speak of a phase of exploration (or of a phase of organization of reasoning, …) when it is the main reasoning mode. The different reasoning modes develop based on different cultural rules, follow different criteria of validity, and may use different semiotic registers. When moving from one mode to another, the student should come to know what is allowed or needed in the new mode. For instance, abductive reasoning is allowed only in Mode 1, and it is not easy for students to move from it to the deductive reasoning needed for the other modes (see Pedemonte, 2007, and Section 2). Similar considerations may concern the use of examples.

From the above description, it is evident that argumentation is an intrinsic component of proving in its different modes of reasoning; moreover, we argue that argumentation (at the meta-level) is a didactical tool that allows the teacher to guide students to manage the different modes of reasoning and the relationships between them in a conscious way. In Section 5 we will present ways to foster both kinds of argumentation.

Lines of argumentation

In a proof, elementary argumentations may form a linear chain in which each conclusion is an input for the following step, thus forming one whole "line of argumentation". However, in many cases argumentation may contain "blocks", or side argumentation branches that meet the main line to input a supplementary data or argument. A block might be considered as a secondary line of argumentation. The hierarchical relations between various argumentations involved in a proof (see Knipping, 2008) may be a source of difficulty for students.

The teacher may use this "lines relating blocks" structure to organize student's proving activity (when necessary) into parts in order that: firstly, they can handle the blocks, and secondly, they can reorganize them according to a main line (see the example below for Pythagoras' Theorem). When planning and managing a collective discussion, the teacher may use this structure as a reference to question locally the arguments according to epistemic rationality, and to link steps according to epistemic and teleological rationality. Globally, the teacher may use this structure to illustrate the pertinence of steps, in accordance with teleological rationality.

5. FROM THEORETICAL CONSIDERATIONS TO CLASSROOM WORK

In this Section we present educational choices related to our theoretical framework and the analysis of some excerpts from teaching experiments that were planned and performed accordingly. The teaching experiments were performed in two middle school classes (age of the students: 12-13 in Italy and 13-14 in France) in spring 2009. Their aims were twofold: as concerns students, we aimed to introduce them to mathematical argumentation and proof and to make them aware of some relevant features of proof; as concerns research, we wanted to ascertain the potential of our theoretical constructs to plan and analyze classroom work.

5.1 The teaching experiments: educational choices

When moving from theoretical framing to educational implementation, we needed to make some general and specific choices regarding the methodology of work in the classroom and the organization of the activities. We now present the educational choices that shaped our teaching experiments, showing their coherency with the theoretical assumptions presented in the previous Sections. As mentioned above, argumentation is crucial both as a part of the proving process and as a means for fostering reflection on the practices of mathematical proof related to different modes of reasoning and to the components of rationality. Consequently, in our teaching experiments we devoted special attention to argumentative activities (at the content level and at the meta-level): giving reasons for the validity of a statement (argumentation with a strong epistemic component), but also unpacking the reasons behind a strategic choice, a way of presenting a solution etc (where also the teleological and communicational components are deeply involved). We hypothesize that argumentation is fostered by an interplay between classroom discussions and individual productions, which can follow each other according to a cyclic organization. Next, we briefly present and discuss our didactical choices, stressing the importance of those cycles of activities.

Mathematical discussion

Mathematical discussions orchestrated by the teacher (Bartolini Bussi, 1996) stimulate efforts of expression and explanation and allow a focus on the three components of rationality, favouring the consciousness of logical rules and their range of validity. Moreover, during the discussion a student may discover others' positions and conceptions and learn how to position herself in regard to these, thus evolving her conceptions and/or processes. Discussing a statement may bring students to methodological and meta - level reflections on issues such as the different role and value of an example in the three reasoning modes (producing an example to support the statement can be an effective step in the exploratory phase, but it is not a valid argument when organising a general mathematical justification). Also, the relation between arguments and the construction of lines of argumentation (mode 2) can be discussed, thus drawing students' attention to the teleological goal of the line of argumentation in relation to its steps.

Individual text production

Based on a Vygotskian perspective, we advocate the importance of text writing in the development of conscious handling of ideas and their organization. Among the individual text productions dealt with in the literature, we consider "narrations of research" (Bonaffé, 1993) useful for exploratory phases, and we advocate "report of what I have learnt after our discussion" to foster learning evolution and consciousness (Assude & Paquelier, 2005). We propose a specific form of individual written production for the learning of proof: the story narration of the proof construction.

Story narration of proof construction

After a classroom discussion, students are asked to write down an individual "story" about the organization of reasoning that was the purpose of the discussion. We hypothesize that this task may lead students to grasp the rationale of the proof, with special attention paid to teleological aspects. The idea of such a task was inspired by the following quotation from Toulmin (1974, p. 6):

> Logic is concerned not with the manner of our inferring, or with questions of technique: its primary business is a retrospective, justificatory one - with the arguments we can put forward afterwards to make good our claim that the conclusions arrived at are acceptable, because justifiable, conclusions.

Writing down a story is also in line with developments in education in recent decades, which recognise the potential of narratives (Bruner, 1990): to narrate a story means to organize facts so as to highlight their (possibly implicit) causal and temporal links; thus, narrative may become a form of thought organization and content understanding (Dettori & Morselli, 2008). We believe it is important to lead students to create a story that connects steps and propositions and fragments of argumentation with reasons of validity that refer explicitly to epistemic warrants. In the story narration, it should become clear that students recognize the involved lines of argumentation, their possible hierarchical relations, and their role in the logical combination that produces the proof. Thus, narration should make clear students' teleological aims. At this level, communication efforts should not yet be subject to production rules for mathematical texts, but instead should respond to the need of mutual understanding.

Interplay between discussion and individual story narration

Following Vygotsky's internalisation principle, we consider that story narration tasks must be combined with classroom discussions according to a cyclic organization: a discussion having a collective construction role should be followed by an individual written story narration having the role of internalisation and personal reorganization of ideas. Finally, a discussion on the produced stories should close the "cycle" by fostering reflection on the crucial elements of a proof and on the strategic choices needed to achieve it.

During early stages of proof learning, story narration should be prepared by suitable tasks to gradually build a specific didactical contract, elicit essential requirements, and develop related competencies. For instance, a narration of a performed exploration may be followed by a discussion starting with teleological matters ("why did you decide to calculate...?"). After students produce their first story narration, the class should compare those texts. Later, the story narration may follow a collective oral organisation of reasons in the style of the expected final text.

The idea of the interplay between mathematical discussion and story narration can be compared to Duval's model of proof teaching (Duval, 1991). He proposes the production of a "proof graph" avoiding linguistic organisation of the propositions, followed by the production of a text based on the interpretation of the graph. His goal is to teach deductive reasoning by drawing the learner's attention to the operative status of propositions (excluding their content). within contrast to Duval's model, our aim is the organization of reasoning (mode 2) based on the content of propositions. In fact, we situate ourselves in a different epistemological perspective of proof construction: Duval's model of deductive reasoning is formal derivation, while for us it is only a model for the final product, not adequate for the school approach to theorems and proof (cf. Thurston's position on the priority of proposition content in proof construction and proof checking: Thurston, 1994).

5.2 The teaching experiments: some examples

Next, we will present three activities that concern important stages of the students' path towards the culture of theorems. All activities combine exploration, conjecture and proof. Our aim is to show how the integrated theoretical framework is used both in the planning and in the a posteriori analysis of classroom activities. The examples also illustrate argumentation at the meta-level.

First example: explorations in elementary number theory

We will show how, within a mathematical discussion, teacher's interventions are crucial in injecting into the debate elements of meta-mathematical argumentation concerning teleological or communicational reasons. The whole sequence was developed over a period of three months and encompassed exploration, conjecture and justification of 8 open problems. For each problem, a cycle of individual or group activities and mathematical discussions was planned. The students mainly encountered situations of cognitive unity: they could, in the proving phase, exploit the arguments that had previously emerged during the conjecturing phase. The aims for students were to build a suitable didactical contract and develop an argumentative attitude. In the first activity introducing algebra as a proving tool, students were given the following task:

> The teacher proposes the following game: Choose a number, double it, add 5, take away the chosen number, add 8, take away 2, take away the chosen number, take away 1. Without knowing the number that you initially chose, is it possible for the teacher to guess the result of the game? If yes, in what way?

The students worked individually, and afterwards they shared and compared their solutions, first in small groups and then within a class discussion. All the groups stated that the teacher can guess the results, because the result is always 10, independent of the chosen number; some groups even tried to find out reasons why the result is always 10, as evidenced by Group B's text:

> **Group B:** With any chosen number, the result is 10 because multiplying by 2 is equivalent to adding twice the chosen number, the same number that after

> must be taken away twice, which gives zero, and doing the other calculations, even in a different order, you always get 10.

In the subsequent class discussion, students were led to shift from answering that the teacher can guess the result because it is always 10, to understanding why the result is always 10. This means, in our perspective, to consciously search for warrants for the conclusion "the teacher can guess the result without knowing the chosen number". During the first group comparison, some students had been able to find a reason why but were not able to communicate the reason to their classmates. Students had realized that solutions in natural language were not always effective in communicating the reasons to the others. This paved the way to the subsequent task, aimed at proposing algebra as a proving tool: "Write the game in the form of an expression, using a different colour for the chosen number. Write an expression that works for any number you choose". The students solved the task individually, and afterwards they shared and compared their solutions in a class discussion. Two main representations of the game were singled out: the representation given by Ric:

$$N*2+5-N+8-2-N-1=10$$

and a sequential representation of the game, by Tor (originally arranged vertically):

$$N*2=A; \quad A+5=B; \quad B-N=C; \quad C+8=D; \quad D-2=E; \quad E-N=F; \quad F-1=10$$

A significant discussion comparing Ric's and Tor's representations of the problem was fostered by the teacher's question: "In your opinion, which of the two representations would be chosen by a mathematician?" The teacher wanted to highlight the importance of strategic choices, such as the choice of the representation, thus emphasizing the teleological dimension. This is an example of argumentation on a meta level, where focus is not on the task itself, but rather on the way of solving the task and the reasons for choosing such a way.

During the discussion, the students shared their motivations for choosing one of the two representations. For instance, Mus's choice ("For me, it is worthwhile to use Ric's representation because it is more schematic and more mathematical") relies on communicational warrants, while Alex's choice ("First of all because [Ric's] follows the text more, and after because it is more correct") relies on intertwined epistemic and communicational warrants. Some students, like Giam, support Tor's representation on the basis of communicative warrants: "Tor's representation, anybody can do it, and he can follow all the steps, while in that of Ric, yes you do it, but you don't really realize what you are doing".

Indeed, both representations are correct from a mathematical point of view (thus meeting the requirements of epistemic rationality) and are also perceived as effective from a communicational point of view. The point is that Ric's expression is more efficient for the original aim of the task (to understand why it is possible to guess the result of the game). This means, in terms of the model, that the Ric's expression better meets the teleological requirements. Reference to the task is a teleological warrant. For the teacher and the external observer, it is important not only that the students choose Ric's representation, but also that they understand the reasons why Ric's expression is more suitable (since it allows us to understand why the result is always 10). The observer's and teacher's interventions during the discussion focus on these reasons:

Observer: you all said a lot of good things, actually doing one or the other is the same, and in both cases you get the result, OK? But do you remember the question of last session? The question was not "what is the result", but "will the teacher be able to guess the result? [...]

Observer: you said: yes, because you always get 10, and some of you also explained something more, we also had some motivations why you always get 10.

Teacher: do you remember? Brac, you said it, because you said that doing NX2 means…

Brac: I mean… it is like doing… yes, it is like doing N+N.

Teacher: N+N. in the expression written by Ric, then… there is NX2, Brac, please go to the blackboard and write N+N under NX2. Do we all agree that it is the same thing? And after you write all the expression: +5-N+8… And you already noticed that… after N+N, what do I have?

Ash: -N.

Voices: two times.

Teacher: and so?

Brac: they all disappear.

Teacher: can I understand this, in Tor's representation?

Voices: no.

Fag: but, at the end there is +8, so, the two representations are equivalent, but Ric's is… easier.

Teacher: but why is it easier?

Giam: because you understand that the chosen number disappears.

Teacher: because I can answer…

Observer: to the original question. Ric's representation helps us understand why it is not necessary to know the chosen number to get the result.

Summing up, we have seen an example of the distinction between argumentation to solve the problem and argumentation on a meta level: the former refers to the search for reasons for the validity of the statement, the latter is an argumentation on the proof itself. The discussion of the strategic choices made (for instance, the choice of a suitable representation to carry out the proof), as well as the discussion on the way of presenting the final proof, are part of such an argumentation. Teleological rationality is emphasized, then communicational rationality intervenes.

Second example: conjecture and proof with no cognitive unity, in geometry

We present this example to discuss elements of argumentation on a meta level that should characterize individual story narration. The students were asked to conjecture about a triangle (say A,B,D are its vertices) inscribed in a circle (diameter [AB]). The exploration, carried out within a DGS environment, was rather poor, but all the students conjectured that the triangle would be right-angled. Afterward, they were asked to prove the conjecture. The information provided by the DGS (measurement of the angles) allowed students to produce the conjecture without any theoretical considerations. In order to prove the conjecture, it was helpful to create a symmetric point to D (the third point of the triangle) and to consider the rectangle completing the triangle. The lack of cognitive unity is due to this supplementary construction. If we analyze the expected proof in terms of blocks and lines of argumentation, we can see three blocks: a preparatory block ("*let us construct the point D', and consider the quadrilateral ADBD'*"), related to the mathematical practice of supplementary or intermediate construction; a second block ("*to prove that the built quadrilateral is a rectangle*"); and a final block ("since the quadrilateral is a rectangle, the triangle is right-angled").

When the students tried to prove the statement, the teacher helped most of the students by suggesting: *"Wouldn't a right-angled triangle be half a rectangle?"* (in French a right-angled triangle is called "triangle rectangle").In this way, the teacher wanted to lead the students to visualize the triangle as part of a rectangle, thus suggesting the supplementary construction. During the individual work, most students found useful arguments, but not ones sufficient to form a proof. Many argued that the triangle was right-angled because the measure of the angle provided by the DGS was 90°. Few students produced a proof like the intended one.

As a starting point for the subsequent mathematical discussion, the teacher selected some arguments previously provided by the students (both incorrect and correct ones) and presented them to the class. For each argument, the class was asked to answer a series of questions: *is it true? Under which conditions? How can we prove it is true? What will this argument be useful for? Is it expressed well? If we need to express it differently, then how?* The aim was to foster the explicit emergence of the three components of rationality. After the discussion, students were asked to narrate the story of the reasoning involved in the proof of the conjecture. Here are two texts:

Text 1:

We produced the following conjecture: it seems that, given that [AB] is a diameter of the circle and D a point on the circle, the angle must be right. First we can say that the angle measures 90°, when we move the circle or when we move D while proving that ABD is always half of a rectangle. To prove this we should draw the symmetric of D (D' on the figure), to prove that the quadrilateral is a rectangle we use the diagonals of the quadrilateral [AB] and [DD']. Because thanks to a property of the rectangle, if the diagonals have the same measure then it is a rectangle. Moreover, we can also say that a rectangle has four angles and that they are all right. So simply, all the angles are right, and the angle is also right. In fact this angle is one of the right angles of the rectangle ADBD', this is why it is 90°.

Text 2:

We want to prove that the angle is right. We can see that the triangle ADB is half of a rectangle, to prove this we tried to take the angle and move it on the circle, then to widen and reduce the circle, concluding the angle remains always 90°. To prove it we draw the symmetric of the point D in relation to O. We can see that [AB] and [DD'] cross each other in their middle and have same length, thus [AB] and [DD'] are diagonals because they are diameters. If we draw [AD'], [BD'] and [DD'] we get a rectangle AD'BD, this quadrilateral has four right angles, and we know that if in a quadrilateral there are four right angles, then it is a rectangle. Conclusion: (no conclusion was written).

Before discussing briefly the two texts, we provide some comments on our criteria of analysis. We were interested in the explicit presence of epistemic reasons, and these were expected to be backed, as in the previous discussion. Communicational and teleological reasons addressed by the teacher during the previous discussion were expected to guide the production of the stories and possibly to be mentioned. Even if present, they were not expected to be backed (in a "normal" mathematical proof text, the teleological choices are not expressed, and certainly not backed!).

In **text 1**, Habermas' components of rationality all seem to be present: they are weaker at the beginning of the text, stronger at the end. The student enchains two arguments of different natures: moving the circle or point D; and proving that the triangle is half a rectangle, thus seeming unaware of the fact that they have different teleological uses at the level of proof creation and conjecturing. In exploration, drawing conclusions from observations is allowed and may be used to produce a conjecture; when producing and organizing arguments, observations cannot be used to draw conclusions, but proving that we have a rectangle can be used. These teleological alternatives reveal the student's epistemic reasons. The fact that her proof seems to rely on the results of DGS exploration, as well as on the reference to mathematical properties, reveals a fragile epistemic rationality. However, she seems conscious of the need for a strategy to prove based on mathematical reference statements (a teleological perspective to organize proof arguments), since she begins her next step by "to prove this...". In the central part, the narration seems to structure the text; teleological reasons guide her when referring to the diagonals. These reasons are rapidly ascertained by the epistemic reason explicitly backing this linking of arguments. Afterward, epistemic reasons allow her to draw a conclusion about the right angles, but the argumentation is not explicit (backing is not exposed). The repetition of arguments and the familiar language used by the student reveal that she is gradually understanding the strategy, and she is conscious that she is now able to make conclusions.

As regards **text 2**, we may say that the student absorbed from the previous discussion the need to make a strategy explicit, but he cannot handle these teleological purposes. The first sentence shows a teleological posture *(We want to prove that the angle is* *right*). The epistemic reasoning is not acceptable in mathematics (the backing is taken from observation and experience). Moreover, the student wants to give a "second proof", unaware of another proving rule (one argumentation is sufficient) and of the uses of epistemic arguments. He is aware of the need to show that constructing D' is a step in the proof, and this again suggests teleological rationality in action. The subsequent sentence ("*we can see that [AB] and [DD']... because they are diameters")* provides a wrong reason of validity for the properties of [AB] and [DD']. The next assertion (AD'BD is a rectangle), which is not justified, is followed by a deductive reasoning relying implicitly upon mathematical backings; afterwards, he turns back from this deduction (it has 4 right angles) to the original assertion (it is a rectangle).

Both texts were selected and given to all students, who were asked to compare them during a class discussion. All the students, despite their difficulties in reading and understanding their classmates' production, recognized many similarities and differences. They preferred **text 1** ("*this is good, she explains it well*"), but nobody addressed the mathematical validity of the arguments or the validity of the mathematical enchainment.

Many students noted that there was no conclusion in **text 2**.

Third example: Pythagoras theorem

Finally, we present a sequence designed to introduce Pythagoras' theorem. During the last school year, the sequence was tested for the first time, and a second teaching experiment is currently underway. Once again, the situation lacks cognitive unity. It is not difficult to get the conjecture through a loosely-guided path, but constructing a proof requires strong guidance by the teacher: changes of frames are needed (from considering lengths to considering surfaces; from geometrical to algebraic register). Moreover, there are side argumentation blocks that must be hierarchically organized within the main argumentation line (and this needs a teleological handling). Teachers' guidance, classroom discussions and story making should allow students to approach the rationale of the proof and offer occasions for learning about proof and proving. To this aim, we planned a series of activities organized into two sequences. The first sequence consists of two different tasks. Its final aim is to conjecture and make sense of Pythagoras' theorem. Task 1 is preliminary: the students are guided to check and justify the triangle inequality, and afterwards they are asked to narrate the story of their reasoning. Its aim is to focus on change of frame (geometric- algebraic), and on the additive relations involved. It is followed by classroom discussion.

> **Task1**: Consider the statement: "In a triangle with sides a, b and c, a+b is always smaller than c". Is it true? always? Why? Prepare yourself to explain how you checked it and why you think it is true, or it is not, or what makes you doubt.

This task also tries to foster exploration through examples, and (especially in the discussion) to lead students to express the rationale of the activity and see the generality of the proposition. Expressions such as "we wanted to see if it is true that...

so we tried to verify it with four examples" are encouraged: such simple narrations reflect an ability to reconstruct the logical skeleton of the activity they went through. This connects a Mode 1 reasoning with a Mode 2, and prepares Task 2.

> **Task 2** (individual): If we consider the squares of the lengths, instead of the lengths themselves, the situation is different. See if a relation between the squares of the lengths of the sides of a triangle exists. Once you think you produced a valid statement (a "conjecture"), put it clearly in words to explain it to your classmates.

Together with the text, the worksheet contains also drawings of right-angled, acute and obtuse triangles, presented with measures of their sides. After the individual solution, students are invited to share and discuss their conjectures within a mathematical discussion. Attention is paid to the different ways of performing the exploration and to the different acceptable expressions of the conjecture(s) (according to mathematical standards). Incomplete or erroneous conjectures may offer fine opportunities to make explicit the important elements of the theorem (in particular, the condition of validity of Pythagoras' theorem, i.e. an angle being right).

> **Task 3** (individual): Write down the conjecture as you now think it should be. Explain it and illustrate it with some examples.

The teacher concludes with the standard formulation of Pythagoras' theorem. This first sequence aims at involving students in Modes of reasoning 1 (drawing, measuring, calculating, …) and 2 (exploiting the gathered data, noticing regularities, expressing a general result, discussing and justifying propositions, and organizing the steps of exploration in relation to a goal).

The second sequence aims at guiding the students to prove Pythagoras' theorem. Given that cognitive unity is not possible, students are guided by means of individual and collective activities through the different blocks of the proof; afterward, they are asked to reconstruct the lines of argumentation.

> **Task 4** (individual): Here we study the proof of the theorem we have conjectured, you will be guided towards this proof. Consider a right angled triangle with sides a, b, c. We use it to build the square A (see below). The area of its central square S is c^2.
>
> 
>
>
> I) Can you describe how A can be obtained by using only our right triangle? explain why S is a square (of area c^2)?
>
> II) Try to write the area of A in two different ways (you may need to arrange the four identical right triangles differently). Find and explain the two ways.
>
> III) How can this help us to validate our conjecture?

Geometrical reasoning is expected to intertwine with algebraic reasoning in order to demonstrate the equality between the areas. If needed, some supplementary tasks can be inserted either for the whole group or for some students.

After the individual work, the teacher orchestrates a mathematical discussion. Attention is paid to the reasoning behind the steps of argumentation and calculation, to the necessity of such steps, and to the connection between geometrical and algebraic arguments. The discussion is intertwined with methodological reflection about the validity of the reasoning, its communicability, and its acceptability by an external reader.

We may notice that the formulation of task 4 is designed to pave the way to the subsequent story narration of the proof. The subsequent discussion is meant to prepare students to write a "story" (task 5).

> **Task 5** (individual): Write down how you organized your steps of reasoning to reach a general justification of the conjecture, and justify why those steps are important.

This task should allow students to grasp and reconstruct the rationale of the proof. After the individual story narration, a selection of stories is provided by the teacher to the whole class and discussed.

6. DISCUSSION AND FURTHER DEVELOPMENTS

The complexity of proving from cognitive, epistemological and educational points of view stresses the need for comprehensive frameworks to analyze students' behavior and guide teachers' classroom activities regarding proving. In recent years, our adaptations of Toulmin's model for argumentation and of Habermas' construct of rational behavior to the case of proving have represented partial contributions in that direction. By integrating the two constructs, we aim at getting a more powerful tool that will address the needs for a more complete analysis of students' behavior and, at the same time, will better frame the role of the teacher as responsible for students' enculturation in the culture of theorems. The integration is based on the idea that those elements of awareness that characterize the expert's rational behavior in proving are inherent in an argumentation at the meta-level (with its specific warrants). This awareness drives and controls the argumentation at the specific level of proving (epistemic validity of arguments, effectiveness of tools and methods, efficiency of communication). Argumentation at the meta-level must be passed over to students through specific educational devices (like the interplay between classroom mathematical discussion and individual story-narration of proof construction). Our examples outline a long-term teaching intervention intended to promote students' enculturation in the culture of theorems (including those aspects of awareness that characterize experts' rational behavior in proving).

The integration, and the resulting experimental activities concerning the crucial phase of the organization of reasoning, suggest further research questions that need to be dealt with in different ways. In particular, on the theoretical side we need:

- to characterize the warrants (and the argumentation at the meta-level) for the other phases of the proving process (the exploration, and the revision of the proof);

- to establish more precise relationships between the warrants of argumentation at the meta-level and the components of Habermas' rationality for the different phases of the proving process;

- to study the relationships between the mastery of meta-mathematical knowledge (in terms of warrants for argumentation at the meta-level) and meta-cognition: indeed, meta-cognition seems to largely depend, in the case of proving activities, on meta-mathematical knowledge.

Further insight into these issues would not only represent an advancement in our theoretical perspectives on the activity of proving, but also would allow us to design better tasks for students and suggest more effective teaching interventions in classroom discussions. In terms of the educational side, the management of the didactical contract in the plan aimed at promoting students' maturation within the meta-level of argumentation reveals an issue that is difficult for teachers to deal with, because focusing on the meta-level of argumentation may "distract" students from the main goal of justifying the validity of statements (a crucial goal not only for proving, but in all kinds of activities that enhance students' intellectual growth).

Finally, preparing teachers to manage the activities illustrated in the previous Section will be a challenge. If appropriately guided by the instructor, proving activities at the adult level might provide an opportunity to see the meta-level of argumentation (through the instructor's guide). Also, analyzing students' performances in ordinary as well in innovative activities of proving might enhance teachers' competencies in identifying students' difficulties and finding ways to overcome them.

7. A REACTION TO BOERO'S *ET AL.* DOCUMENT

Our research lies in student argumentation during activity carried out to formulate a conjecture, find reasons it must be true, and construct a proof in similar ways as experts would. Through the following analysis of an episode within an experiment involving pre-service math teachers, we aim to provide elements for the study of the utility of the integrated model. Specifically, we want to illustrate two aspects: some imprecision in the description of Habermas' model which, we think, requires more elaboration, and the difficulties in using the integrated model when analyzing ongoing argumentation. The students were in their third undergraduate geometry course when the experiment was carried out. This course and the previous one had three distinguishing characteristics: the axiomatic system used was collectively constructed by the group through an inquiry community of practice, the theoretical elements of the system arose from the conjectures students formulated as solutions to specially designed situations, and a dynamic geometry program was used as a mediation tool for the process of learning to prove within that axiomatic system. At one point, the students were given the following problem:

> With Cabri, construct a circle with center C and a fixed point P in its interior. For which chord AB of the circle, that contains point P, is the product AP × PB maximum?

They modelled the situation, dragged and discovered that the product does not vary. We use the integrated model to analyze an excerpt of the interaction within a group of three students when they formulated and outlined the path to construct a proof for their conjecture: *Given a circle with center C, a fixed point P which belongs to the interior and a given chord AB which contains P, then the product AP × BP is constant*. Once the conjecture was written, Nancy suggested substituting the phrase "a given chord AB" by "any chord AB". This incident brings forth the communicative aspect of the Habermas model: we see a conscious use of the universal quantifier when they change a particular statement to a general one, and thus an adherence to rules established for the formulation of conjectures. Later in the intervention [line 193 below], Nancy's reference to the word "any" seems to acquire an epistemic sense because through it she manifests that obtaining an equal product for the second chord validates the claimed invariance. This and our analysis of excerpts where our students discuss whether words such as "let", "exist", "determine", "localize" or "choose" are the adequate ones to use when expressing an idea leads us to ask: Is this a part of the communicative or epistemic aspect? Is using schemes to organize a proof, such as two columns, considered a communicative aspect of an argument?

162 Alejandro:	In the proof we can construct another chord, right? [Assenting murmurs.] To have similar triangles.
168 Fabián:	In every case we get similarity? Or can we achieve congruency at some point, with two chords?
169 Nancy:	It would be better to use congruency to prove it is always equal.
170 Alejandro:	No, because…what we need to prove is a ratio.
171 Fabián:	Yes, a product.
172 Obs:	The what?
173 Alejandro:	The ratio between [Moving index finger over the chord in the calculator construction.] segments AP and PB.
174 Fabián:	Then better similarity.
176 Alejandro:	Because the only moment in which they can be congruent is when P is the center.
177 Nancy:	Also, yes.
178 Fabián:	That is why I ask if in some point they would be congruent but then when they are congruent they will be similar.
179 Nancy:	Yes, it's the same.
180 Fabián:	Anyway it's the same similar as… but it is better similar than congruent.
181 Obs:	I want to ask a question: you Alejandro said something about creating another

chord? … And why did you think of that?

186 Alejandro: Because since we see the product will always be the same, right? [With index finger points to the constructed chord on the screen.] Then another chord [points to a chord that hasn't been constructed.] can give us similarity or ratio between this side, this segment that we would create new and this [Points to the imaginary chord..], that… [Starts constructing the other chord.], wait, we'll create it

190 Alejandro: [Constructing the other chord on the computer.] This point… on the circle… then, here we create similar triangles [moves index sketching a triangle on the screen.]

191 Nancy: We would have that the ratio of … or that by ratios we get that AP times BP is going to be the same for both chords …

192 Fabián: It will always be the same.

193 Nancy: And that way we would confirm that it would be for **any** chord [Pause.]

195 Nancy: Then it wouldn't be only for the one we use; we also compare it with another one.

The students have set their goal: proving that the product is constant for any chord. This leads them to construct two chords that contain P with the purpose of obtaining similar triangles, which allow them to work with ratios that will lead to equal products. This goal motivates an auxiliary construction without which it is practically impossible to prove the conjecture. Therefore, we recognize the teleological aspect in their argumentation because they sketch a plan to reach their goal and propose an auxiliary construction as a tool to obtain it. They have at the same time evoked similar triangles as the idea within which they feel sure to find the crucial step of their proof. This leads all their latter actions in a process towards constructing the justification, and we identify epistemic rationality in these actions. These observations suggest that the difference between the teleological and epistemic aspects is that the first one lies in simply suggesting a path and the latter in justifying the choice. We ask ourselves two questions: Besides referring to the conscious and appropriate validation of the propositions, does the epistemic aspect include considerations about the theoretical status of the statements mentioned? In which of the aforementioned rationalities do we set the identification of the theoretical field that is used to sketch the plan for a proof? We have two different ways of interpreting Toulmin's model in this excerpt. For the first one, we recognize that using Toulmin's model requires that the three elements of the argument be explicit. Due to the fact that this is an ongoing argument, we think it is possible to take the liberty to see the process as follows: asking whether another chord can be constructed in the proof is asking whether it is useful for the proof. In that sense, it is the expression of missing data and therefore of abductive reasoning. Although the warrant is not given explicitly, the claim (170) is equal ratios. Being able to convert this argument into a deductive one is difficult because the data is not a statement but an idea for an auxiliary construction. This is why, when the students report to the observer how they thought of constructing another chord, the deductive argument uses three other elements: having similarity (data) gives ratios and, through these, products (claim). The problem is that there is no warrant. If we look at this situation in a more rigorous form, we can say that they are evoking theory and not constructing an argument. Another possible analysis is that, initially, they seem to assert that the two chords that contain *P* determine similar triangles (claim) [162]; no warrant is given, and instead the generality of the statement is questioned (rebuttal) [168]. Up to that point, a mathematical statement is formulated but no argumentation given. Nancy's contributions detour the conversation because, in fragment [169-180], she refers to the relevancy of similar/congruent triangles for the proof. Taking into account the discovered invariance of the product (data), it is asserted/proposed that the use of similar triangles will allow completion of the proof (claim) because, from it, equal ratios can be deduced and from these equal products (warrant). The following questions arise: what is the structure of this argument? Can this argument be outlined by Toulmin's model? Would it be of interest to outline arguments such as the one presented? With respect to the use of Toulmin's model, would it be convenient to specify with greater precision the types of arguments that can be modelled? Does the difficulty in using the model lie in the fact that the students are simultaneously trying to find what is pertinent and trying to construct a justification?

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