MENTAL MODELS AND THE DEVELOPMENT OF GEOMETRIC PROOF COMPETENCY

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We propose a cognitive model of geometric proof competency based on ideas from psychological research on deductive reasoning and the idea of figural concepts (Fischbein, 1993). We describe two possible mechanisms contributing to the development of geometric proof competency: deductive reasoning ability and perceptual chunks. Data from a longitudinal study indicates the particular influence of perceptual chunks.

INTRODUCTION

Learning to construct mathematical proofs is regarded an important objective of mathematics instruction in many countries (e.g., NCTM, 2000). Moreover, in most countries, proofs are presented within elementary Euclidian Geometry. Accordingly, this contribution considers geometric proof competency from a cognitive point of view, aiming at a description of its development in lower secondary school. Following Weinert (2001), competencies are defined as cognitive abilities and skills, which individuals have or which can be learned by them. These abilities and skills enable them to solve particular problems and encompass the motivational, volitional, and social readiness and capacity to utilize the solutions successfully and responsibly in variable situations.

In this sense, our interest is not to explicitly describe the students' *understanding* of mathematical proof, but to model their competency to construct geometry proofs. We hold a normative point of view on mathematical proof which is influenced by curricula for secondary school regarding the common basis of accepted mathematical statements and also includes criteria for the acceptance of students' proofs as mathematically correct.

MODELING GEOMETRIC PROOF COMPETENCY

There are approaches to model students' competence to construct geometry proofs from a cognitive perspective. Duval (1991) distinguishes two levels of proof construction: The organisation of propositions into a deductive step and the arrangement of deductive steps into a proof. This distinction is reflected in a three-level model of geometric proof competence proposed and empirically validated by (Heinze, Reiss, & Rudolph, 2005): Level I encompasses geometric calculation problems which do not require any general deductive arguments, level II corresponds

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to solving problems which require the construction of single-step proofs, and level III includes proofs requiring more than one step.

Duval's (1991) ideas build upon a view of deductive reasoning as manipulation of propositions with regard to certain formal rules. There are psychological theories that describe deductive reasoning in this sense. Alternative theories exist that have a similar or better empirical basis (see Bara, Bucciarelli, & Lombardo, 2001, for an overview). In this contribution, we do not address the question which theory is more adequate to describe deductive reasoning in general, but try to apply an alternative theory – the Mental Model Theory (Johnson-Laird, Byrne, & Schaeken, 1992) – to geometry proofs. Our aim is to provide a framework for describing the development of geometric proof competency as well as several of students' problems that can be observed.

The Mental Model Theory and geometric proof competency

According to the Mental Models Theory (MMT), deductive reasoning is achieved by building mental models of the premises, i.e. mental representations – which usually have a figural component – that represent a set of "states of affairs" in the real world, which are compatible with the premises. Depending on the kind of reasoning, these models are conceptualized in different ways. Bara, Bucciarelli, and Lombardo (2001) propose a conceptualization that applies to a wide variety of reasoning types. For geometry proof, we will propose a different conceptualization.

Deductive reasoning according to the MMT consists of three phases (Johnson-Laird, Byrne, & Schaeken, 1992). Given a set of premises, the individual builds a mental model representing a set of states of affairs that are compatible with the premises (e.g. a representation of a set of examples). Secondly, the individual draws a provisional conclusion based on the model. This conclusion reflects properties of the model which are not directly accessible from the premises. Finally, the provisional conclusion is validated by checking it against alternative models of the premises. If no model can be found that is incompatible with the provisional conclusion, it is accepted and integrated into the model, otherwise it is rejected (and the process starts again).

With regard to mathematical reasoning we propose an extension of the MMT by a fourth phase. The specific nature of deductive arguments in mathematics is to ensure that indeed no alternative models can exist which render the conclusion false. This is usually done by searching for some (accepted) theorem that eliminates the possibility of such incompatible alternative models. The central idea of MMT is that deductive arguments are found using inductive processes (based on a single model) which are afterwards checked against alternative models and only then secured by searching for an appropriate theorem.

For single-step proof problems, one may get the impression that the first three phases of the reasoning process are not necessary. If the student performs only the one step necessary to prove the hypothesis (i.e. recalls the necessary theorem), this is true.

Otherwise the problem-solving process of searching a chain of deductive arguments starts in the same way as for multi-step proofs – with possible dead ends. Nevertheless, this lower complexity is indeed one of the specifics of single-step proof problems.

Mental models for geometry proof

To make predictions based on MMT, it is necessary to conceptualize the term "Mental Model" for the context of geometry proof. Mental models can be formed by encoding verbal information or as a result of perception (Johnson-Laird, Byrne, & Schaeken, 1992). In most geometry proof problems, the premises are usually given as verbal or symbolic information together with a geometric figure[1]. Thus the mental model has to integrate two kinds of information: visual information and conceptual information reflecting the premises given explicitly. Fischbein (1993) discusses mental representations of this kind under the term "figural concepts". Figural concepts are understood as mental representations having figural and conceptual character, but which are not reducible to one the two aspects without loss of information. Fischbein refers to figures intrinsically controlled by conceptual constraints. For example, the mental visual image of an isosceles triangle contains less information than the figural concept that has the information of equal sides attached to it. It is possible to operate mentally with the figural representation, keeping track of the conceptual constraints. According to Fischbein (1993), an ideal case would be a figural representation totally controlled by the conceptual constraints. Nevertheless, he points out that the figural representation is subject to Gestalt forces that can support or impede problem solving processes.

When considering multi-step proof problems, reasoning is a problem-solving process which is directed towards the hypothesis given in the problem formulation. This additional information is relevant during the reasoning process because it values certain possible intermediate conclusions from a given model higher than others. The conclusions that are considered "closer" to the hypothesis will be preferred. Thus the hypothesis should be considered as a part of the mental model. Here we follow Duval (1991) with the idea that the *status* of an information must be considered, distinguishing the hypothesis from other conceptual information in the model.

DEVELOPMENT OF GEOMETRIC PROOF COMPETENCY

When analysing the development of geometric proof competence, different perspectives can be taken into account. Küchemann and Hoyles (2006) for example gave evidence that individual development need not result in a monotonous increase of performance in solving the same proof problem over several years. From a perspective on the system level using appropriate tests consisting of several proof items which meet the requirements of psychometric models, we can expect an

[1]If no figure is given, constructing a figure compatible with the premises is usually an important strategy.

increase in students' performance. Nevertheless, it is not possible to describe specific individual learning trajectories. Our approach here is to analyse development on the system level using quantitative methods.

Proof competency is a complex construct, and its development can occur in several ways. An increase can be due to a better ability to reason deductively in the sense of the four phases described above, or due to an increased ability to generate, manipulate and check mental models. Moreover, an increase in geometric content knowledge may facilitate an improvement in both fields. We will give two examples.

Firstly, an increase in deductive reasoning ability may occur from an improved capability of coordinating the problem-solving process of searching and selecting possible arguments. This includes, for example, the ability to generate and keep track of intermediate hypotheses in the mental model that are proven in a subordinate proof problem, before the intermediate hypothesis is integrated into the model. In order to facilitate this, the student must be able to deal with information of different status (Duval, 1991) within the same mental model. Moreover, general problem-solving skills and meta-cognitive competencies have an influence on this ability to coordinate the reasoning process.

On the other hand, improvement in geometric proof competency may occur by a reduction of the complexity of the proving process: Koedinger and Anderson (1990) describe how *perceptual chunks*, i.e. mental associations between certain prototypical figural configurations and mathematical concepts that apply to them can support the proving process. If a student matches a sub-figure of his/her mental model with a configuration corresponding to one of his/her individual perceptual chunks (e.g. a pair of vertical angles), the result of a corresponding proof step (i.e. the conclusion that the angles are congruent) can be integrated into the mental model immediately without generating and checking alternative models first. If the corresponding theorem (the vertical angles theorem) is also part of the chunk, the corresponding proof step does not have to be constructed within the problem-solving process. This reduction of complexity is of particular importance, if the number of proof steps to be constructed is reduced to one – making the item change the from competence level II to level III in the model of Heinze, Reiss, and Rudolph (2005) for this single student. We can assume that students learn perceptual chunks during problem solving in their geometry instruction.

We will focus on these two aspects of development, keeping in mind that different processes may be at work additionally, e.g. improved availability of problem schemata (Koedinger & Anderson, 1990).

RESULTS FROM A LONGITUDINAL STUDY

To contribute to the question how the two processes in the theoretical model described above interact, we will present data from a longitudinal study on geometric proof competency we conducted with students from grade 7, 8, and 9 (13–15 years old)[2]. The sample consisted of N=196 students that took part in all three measurements are taken into account in our analysis. All students visit the high-attaining school track “Gymnasium” in Munich (Germany). Achievement tests with 10–12 open-ended items were administered each year within one 45-minute lesson by trained assistants.

Test instruments and psychometric model

The tests were constructed with respect to the competence model proposed by Heinze, Reiss, and Rudolph (2005), using items from all three levels. We started with this simple model considering the number of deductive steps necessary for a proof, but we also took into account that the complexity of a proof item may be smaller due to the availability of perceptual chunks. This means that the allocation of items to the three levels may differ between the different grades. At this point we benefit from the fact, that our sample was recruited from German gymnasiums, i.e. the high attaining school track that is visited by about 30% of the students. We can assume that this sample is quite homogenous with regard to mathematical knowledge, so that the analysis of perceptual chunks can be expected at a certain grade. One example of an item assigned to level III in grade 7 and to level II in grade 9 is given in figure 1. It was solved by about 10% of the students in grade 7 which corresponds well to the assignment to level III. In grade 9 about a third of the students succeeded on this item. We assume that for 9th graders the sub-figure corresponding to a pair of vertical angles is part of a perceptual chunk which leads to an integration of the information about the equal angles as described above even before the proving process starts. In this case, only one deductive step remains: The new angle and the other two lie at a straight line, thus add up to 180°.

Prove $\alpha + \beta + \gamma = 180°$

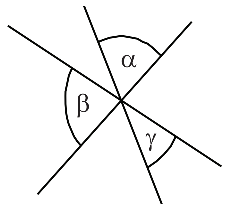

Figure 1: Item in grade 7/9

Our statistical approach to describe the development of geometric proof competency in spite of different tests used in different years is to model the empirical data using the dichotomous unidimensional Rasch Model (Rasch, 1960)[3]. Checks of item fit values for each item provided satisfying results, indicating that the model is appropriate. The use of probabilistic test theory makes it possible to model individual regressive performance on single items as reported by Küchemann and Hoyles (2006). On the other hand, probabilistic modelling makes it difficult to interpret the competency of a single student. Hence, all results reported here are on the system level. The three tests were connected by nine anchor items. Four of these anchor items were assigned to different competency levels in different grades, like the item in figure 1.

[2] By July 2009, data from a fourth point of measurement in grade 10 will be available.

[3] The scaling was performed with the Software *ConQest* (Wu, Adams, & Wilson, 2006) using WLE estimates.

Quantitative development from grade 7 to grade 9

Since ability parameters from the three measurements are aligned on a common scale using the Rasch Model, a repeated measures ANOVA was used to analyse the development. The effect for time of measurement was significant ($F(2,390) = 35.06$, $p < 0.001$, $\eta^2 = 0.154$). This indicates that geometric proof competency changed between grade 7 and grade 9. Mean values, standards deviations, effect sizes, and data of t-tests are given in table 1, showing an increase in both periods of time.

The effect sizes are comparable to the growth of mathematics achievement within one school year in Germany (e.g., Prenzel et al., 2006). The increase is larger in the first period which can be ascribed to instruction in geometry proof at the beginning of grade 8 and the fact that the tests were based on contents from grades 7 and 8.

Splitting the sample into thirds with respect to their proof score in grade 7, the influence of prior knowledge on the development can be studied. Results of t-Tests showed significant increases in the first period for the lower and middle third and in the second period only for the middle third. The group with highest prior knowledge could not increase its performance significantly between grade 7 and 9[4]. As found in previous studies, only students from this group were able to solve a considerable number of multi-step items in grade 7 (Heinze, Reiss, & Rudolph, 2005). On the average, they also solved about 90% of the single-step items correctly. The fact that these students are not able to improve their performance indicates that the ability to find multi-step proofs does not increase for these students.

	M	SD	d	p (t-test)
grade 7	93.7	7.09	0.37	<0.001
grade 8	96.3	7.22		
grade 9	98.3	7.67	0.27	<0.001

Table 1: Quantitative results

competence level I (%)	competence level II (%)	competence level III (%)
76.6	55.8	23.4
74.7	50.1	18.5
78.7	42.9	15.8

Table 2: Qualitative results

Qualitative development from grade 7 to grade 9

The data described above shows that students are able to solve more complex proof problems in higher grades. We tried to analyse which of the two development processes derived from the model above is more plausible to cause this effect. In any case, a student in grade 9 is expected to solve more, and more complex proof problems than a student in grade 7. But if this better performance is mostly due to an individual mental reduction of complexity by the use of perceptual chunks, it can be questioned if this is a desirable increase of proof competency. From a mathematics education point of view the increase of deductive reasoning ability should also be an

[4] We emphasize that no ceiling effects occurred in the tests at all times of measurement.

outcome of proof instruction. Table 2 contains mean solution rates for the sub-tests corresponding to the three competency levels in each grade. Here items were assigned to different levels in different grades, if a reduction of complexity by perceptual chunking processes could be expected from our knowledge of very common geometry concepts in the corresponding grades. Here, hardly any improvement in the performance on those multi-step proofs, which cannot be reduced mentally in their complexity, can be observed.

These data support the assumption that the observed quantitative increase in geometric proof competency is primarily *not* due to improvement in deductive reasoning capability, but rather to an improvement in building mental models efficiently, reducing the complexity of the proof problems.

DISCUSSION

The presented data indicate that improvement in geometric proof competency may not be primarily caused by better reasoning skills, but rather by a better quality of geometry knowledge, particularly by the availability of perceptual chunks. Our interpretation of these findings supports Duval's (1991) idea that the number of proof steps is an important predictor of the difficulty of a proof problem. Nevertheless, we assume that additional effects may lead to a reduction of complexity on an individual level, rendering multi-step proof problems into single-step problems due to their cognitive representation.

On the theoretical side, an explanation is provided by the proposed adaptation of the MMT, if the idea of figural concepts is used as a basis to conceptualize mental models for geometry proof. This makes it possible to integrate the work done by Koedinger and Anderson (1990) about perceptual chunks into a common framework with a psychological theory of deductive reasoning. Syntactic theories based on the manipulation of verbal or symbolic propositions would not be adequate for this integration. Particularly regarding geometry proof, the model proposed here provides additional explanative power regarding some effects regularly observed in research on proof: Inductive proof schemes (Harel & Sowder, 1998), for example, can be understood as omitting the validation phase three (and phase four) in the reasoning model. Circular arguments can occur because of an invalid integration of the hypothesis into the model without considering its special status.

Implications of MMT in connection with mathematical proof in general have been discussed by Stylianides and Stylianides (2007). With respect to geometry proof, the idea of figural concepts as mental models deserves special attention and provides new ideas to foster geometric proof competency by supporting students in generating, manipulating and checking mental models efficiently. In view of our results, the ability to deal with the specific complexity of multi-step proofs deserves special attention. One first approach that can be interpreted as providing a technique to foster the construction of mental models is the reading-and-coloring strategy studied by Cheng and Lin (2005).

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