HIGHSCHOOL TEACHERS' KNOWLEDGE ABOUT ELEMENTARY NUMBER THEORY PROOFS CONSTRUCTED BY STUDENTS

Ruthi Barkai, Michal Tabach, Dina Tirosh, Pessia Tsamir, and Tommy Dreyfus

Tel Aviv University[1]

This study investigates changes in teachers' knowledge regarding their students' construction of correct and incorrect proofs within the context of elementary number theory, before and after a professional development course. Twenty high school teachers were requested twice to suggest correct and incorrect proofs their students might construct, at the beginning and at the end of the 15 week long course. The suggested proofs were analysed according to modes of representation. Results indicate that the teachers' suggestions of correct and incorrect proofs students might construct increased both in number and variety.

THEORETICAL BACKGROUND

Recent reforms in mathematics education recommend including proofs as a key component in school mathematics (Australian Educational Council, 1991; Israeli Ministry of Education, 2004; National Council of Teachers of Mathematics, 2000), because of the vital role proofs play for validation and for refutation in mathematics (Aigner & Ziegler, 1998; Thurston, 1994). Different types of proofs require the use of different methods. For a universal statement a general proof, covering all relevant cases is necessary to validate the statement while a single counter example is sufficient to refute such a statement. For an existential statement a single supportive example is sufficient to prove the statement, while a general proof, covering all relevant cases, is necessary to refute the statement. In addition, a proof may be given in various modes of argument representation (Stylianides, 2007), such as verbal, numeric or symbolic representation.

Teachers are responsible to incorporate proofs and proving in everyday school practise. What are the types of knowledge a teacher needs to implement proofs in his class? "Teachers' subject matter conceptions have a significant impact on their instructional practices" (Knuth, 2002, p. 63), and hence, analysing teachers' knowledge with respect to proof and proving is important. Knowing to verify valid statements and to refute invalid ones is an essential component of teachers' subject matter knowledge. Yet, it is not the only requirement. Hill, Ball, Sleep and Lewis (2007) rhetorically ask "Is the knowledge mastered by someone who majors in mathematics sufficient content knowledge for teaching?" (p. 112). Teachers need to be able to evaluate students' suggested proofs to various statements. In addition, teachers need to be familiar with the ways students correctly and incorrectly justify a variety of statements. This latter type of knowledge is the focus of the present study.

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What does the literature tells us about teachers' familiarity with the ways students justify statements? Not much. We found several studies relating to secondary school teachers' evaluation of students' given justifications (e.g., Dreyfus, 2000; Healy & Hoyles, 2000). However, we did not find studies in which teachers were asked to provide correct and incorrect justifications that students are likely to construct for various statements.

Hence, we looked for studies related to students' knowledge of proofs and proving. Studies have shown that students are not always aware of the necessity for a general, covering proof when proving the validity of a universal statement for an infinite number of cases (e.g., Bell, 1976). Healy and Hoyles (1998, 2000) found that 14-15 year olds have difficulties constructing a complete proof based on deductive reasoning. Balacheff (1991) found that students relate to counter examples as bizarre instances and do not always recognize a counter example as being sufficient to refute a universal statement. Regarding the types of representations used by students when constructing proofs, Bell (1976) found that none of the 36 high school students in his study used an algebraic proof when proving a numerical, universal conjecture. Healy and Hoyles (1998, 2000) found that students preferred verbal explanations over other kinds of representations.

In the present study we ask two questions: (1) Are high school teachers familiar with correct and incorrect justifications that students may construct for various elementary number theory statements? (2) To what extent did a professional development course contribute to teachers' knowledge of students' justifications?

SETTING AND METHOD

Twenty high school teachers participated in the study. All of them were studying towards a master degree in mathematics education.

The participants participated in weekly meetings, two hours long, of a 15 week professional development course. The course aimed at enhancing participants' knowledge with respect to mathematical aspects of proofs and proving, as well as with respect to didactical aspects of teaching proofs in secondary school. The course included the design, by the participants of a learning unit (2-4 lessons), in the domain of elementary number theory (ENT). The ENT context was chosen as the related concepts were thought to be familiar to teachers and to middle grade students, enabling the teachers to focus on proving the statements and minimizing difficulties that may have arisen due to problems with the content domain and misunderstanding of terminology. The participants implemented the unit they designed in their own classes, and reported back on these implementations during the course sessions.

Tools

At the beginning of the course, a set of three questionnaires, consisting of six ENT statements was administered to all participants (Table 1). The validity of each statement is determined by a combination of the predicate (always true, sometimes true, or never true) and the quantifier (universal or existential). The teachers were asked to answer a three part questionnaire: first to produce a proof (or a refutation) for each of the six statements, which they consistently did correctly (part 1). Next, for each of the six statements, teachers were asked to suggest as many correct and incorrect proofs that, in their opinion, students would give for these statements (part 2). Finally, the teachers were presented with different, correct and incorrect justifications for each statement (part 3). In this paper we limit the discussion to teachers' answers to the second part. (A discussion on the findings relating to part 3 can be found in Tsamir, Tirosh, Dreyfus, Barkai and Tabach, 2008).

Predicate Quantifier	Always true	Sometimes true	Never true
Universal	**S1**: The sum of any 5 consecutive natural numbers is divisible by 5. *True*	**S2**: The sum of any 3 consecutive natural numbers is divisible by 6. *False*	**S3**: The sum of any 4 consecutive natural numbers is divisible by 4. *False*
Existential	**S4**: There exists a sum of 5 consecutive natural numbers that is divisible by 5. *True*	**S5**: There exists a sum of 3 consecutive natural numbers that is divisible by 6. *True*	**S6**: There exists a sum of 4 consecutive natural numbers that is divisible by 4. *False*

Table 1: Classification of statements

Data Analysis

All proofs presented by the teachers were categorized according to their modes of representation (Stylianides, 2007). This analysis resulted in three modes of representation: numeric, symbolic, and verbal.

RESULTS

We first present overall results about correct and incorrect justifications that the teachers suggested as students' constructs. We follow with an analysis of modes of argument representation for the correct justifications with examples of teachers' suggestions, and finally a similar analysis with examples for the incorrect justifications.

At the beginning of the course, our participants suggested a total of 291 correct and incorrect justifications that students may give to the six statements in Table 1. At the end of the course, the total number of justifications had increased by 51%, resulting in 440 justifications. At the beginning of the course, the teachers suggested 169 correct and 122 incorrect justifications and at the end of the course 255 correct and 185 incorrect justifications. Tables 2 and 3 present the numbers of correct and incorrect justifications (respectively) suggested by the teachers as possible justifications that students will construct to each statement. Overall, it seems that they found it easier to think of correct justifications that students will construct than of incorrect justifications. A more detailed examination of the tables reveals, however, that this is the case only for statements S2-S5; for statements S1 (universal, always true) and S6 (existential, never true), on the other hand, the teachers provided more incorrect justifications then correct justifications before the course. These two statements require a general-cover proof. This is in line with research findings about students difficulties with general-cover proofs (Bell, 1976). However after the course, the number of correct and incorrect justifications in the case of these two statements was almost the same.

	S1	S2	S3	S4	S5	S6	
	---------- U n i v e r s a l		----------	-------- E x i s t e n t i a l		--------	
True	(Always)	(Sometimes)	(Never)	(Always)	(Sometimes)	(Never)	Total
Before	25	33	33	31	31	16	169
After	47	50	44	42	42	30	255

Table 2: No. of justifications teachers provided as students' **correct** justifications

The number of suggestions for incorrect justifications to statements S3 (universal, never true) and S4 (existential, always true) was significantly lower than the number of suggestions for the other statements, both before and after the course. The teachers' common remark regarding S3 was that since any numerical example of the sum of four consecutive numbers is not divisible by four, students will not err in this case. Similarly, with respect to S4, teachers claimed that since the sum of any five natural consecutive numbers is divisible by 5, students will not err in this case either. Teachers' claims in these cases are in line with reports about students' tendency to start with examples (Healy & Hoyles, 1998).

	S1	S2	S3	S4	S5	S6	
	---------- U n i v e r s a l		----------	------ E x i s t e n t i a l		------	
True	(Always)	(Sometimes)	(Never)	(Always)	(Sometimes)	(Never)	Total
Before	29	33	10	5	24	24	122
After	46	43	17	9	37	33	185

Table 3: No. of justifications teachers provided as students' **incorrect** justifications

It is notable that the change from pre test to post test in the number of justifications provided for each statement varied considerably. The largest change was found in correct justifications for statements S1 and S6, which means that the participants could provide more correct general-cover proofs after the course. Also, the participants could provide more incorrect proofs for statements S3 and S4.

We now turn to the modes of argument representation suggested by the teachers for the correct justifications of each statement before and after the course (Table 4).

	S1		S2		S3		S4		S5		S6	
	---------U n i v e r s a l---------						--------E x i s t e n t i a l--------					
True	(Always)		(Sometimes)		(Never)		(Always)		(Sometimes)		(Never)	
	Before	After	Before	After	Before	After	Before	After	Before	After	Before	After
Symbolic	23	31	15	24	14	21	13	19	9	17	15	23
Verbal	2	16	0	3	0	4	0	0	0	0	1	7
Numeric	0	0	18	23	19	19	18	23	22	25	0	0

Table 4: Categorization of the correct justifications that teachers suggested as students' justifications according to the mode of argument representation

In each mode of argument representation we can see an increase in the number of suggested correct justifications. After the course the teachers suggested more correct symbolic justifications; all the participants provided the symbolic justification where x represents the first of the consecutive natural numbers for each statement, including statements S2 – S5, for which supportive or counter examples suffice as proof. A few other symbolic justifications were suggested – mainly after the course, representing the middle element as x, using mathematical induction, or using the formula for the sum of an arithmetic sequence. Many more verbal justifications were presented after the course than before (30 vs. 3), especially for statements S1 and S6, which required a general-cover proof. An example for a verbal justification to S1 before the course: "*One of five consecutive numbers is divisible by five. The other numbers, when divided by five, will give the remainders 1, 2, 3, 4. The sum of these remainders is ten, which is divisible by five*". After the course, additional justifications were given, many of them were of the kind: "*I will check the sum of the first five consecutive numbers: 1 + 2 + 3 + 4 + 5 = 15, divisible by five. The sum of the next five consecutive numbers can be obtained by adding five to this sum (since each addend is larger by 1, hence the sum is larger by five). In each step we add five to a number that is already divisible by five; hence the sum is always divisible by five. The statement is true*". While twenty correct verbal induction type justifications were presented after the course, only seven induction justifications were presented symbolically. With respect to numerical justifications, before the course the teachers provided justifications which included one numerical example – either as a supportive example or as a counter example. After the course, there was a general small increase, especially in the category of several numerical examples (for example, the student will choose a supportive example, and also a counter example as justification). This is in line with research findings that indicate students' tendency to use more than one example as justification (Bell, 1976).

The modes of argument representation in the case of incorrect justifications provided by the teachers as their students' construction can be examined in Table 5.

	S1		S2		S3		S4		S5		S6	
	---------U n i v e r s a l---------						--------E x i s t e n t i a l--------					
True	(Always)		(Sometimes)		(Never)		(Always)		(Sometimes)		(Never)	
	Before	After	Before	After	Before	After	Before	After	Before	After	Before	After
Symbolic	4	8	7	12	6	7	2	6	10	12	4	6
Verbal	4	1	2	1	4	3	2	1	2	1	3	2
Numeric	21	37	24	30	0	7	1	3	12	24	17	25

Table 5: Categorization of the incorrect justifications that teachers suggested as students' justifications according to the mode of argument representation

For the incorrect justifications, as for the correct ones, we note an increase in the number of the symbolic justifications for each statement. It seems that the participants expand their repertoire of students' errors. Two categories of symbolic lapses were identified in teachers suggested justifications. Generality lapses related to cases were the symbolic representation of the consecutive numbers was wrong, like *x*, *x*, .. or *1x*, *2x* …, and hence the generality of the justification was violated; and inference lapses like an invalid chain of inferences ("*The sum of five consecutive numbers: x + x+1 + x+2 + x+3 + x+4 = 5x + 10, 5x = -10, x=-2. There is a solution and hence the statement is true*"). In some cases the wrong inference was to transform an expression into an equation, while in others the expression that represents the sum of the consecutive numbers was given a wrong interpretation. A decrease was found in the verbal mode of argumentation for all cases of incorrect justification. This is not in line with the increase evident for the correct justifications (Table 4). It seems that intuitive verbal justifications (like, "*The sum of any five consecutive natural numbers is divisible by five, hence the sum of any four* *consecutive natural numbers is divisible by four*"), was abandoned by the participants. With respect to the numerical mode of argumentation, we note an increase for each statement. The teachers seem to have become aware of a variety of errors with respect to numerical examples as justifications, like providing several examples and concluding that a statement holds for all cases, or checking a "small" numerical example and a "large" numerical example, and concluding that the statement holds for all cases. This is in line with research findings about students' incorrect justifications (Harel & Sowder, 2007).

CONCLUDING REMARKS

In the present study we asked two questions: (1) Are high school teachers familiar with correct and incorrect justifications that students may construct for various elementary number theory statements? (2) To what extent did a professional development course contribute to teachers' knowledge of students' justifications?

As the range of teachers’ suggestions for correct and incorrect justifications suggests, the participants in our study think that students' use of the symbolic mode of argument representation is extensive. Also, it seems that at least some of the participants showed no awareness of students' preference for the verbal mode of argument representation. While the participants' tendency to suggest symbolic justifications did not decrease after the course, their suggestions for verbal correct justifications in the case of general cover proofs increased considerably.

The professional development course influenced both, the number and variety of high school teachers' responses when asked to present students’ correct and incorrect justifications to six ENT statements. In particular, we point to the increases in the number of correct verbal justifications and the number of incorrect numerical justifications; indeed, the literature indicates that students tend to use verbal justifications for proving correct ENT statements, and numerical examples when constructing an incorrect justification (Harel & Sowder, 2007). The participants also expanded their repertoire of symbolic lapses, which may help them identify such justifications when presented in their own classes.

Three factors may have contributed to the observed changes: The professional development course itself has probably had an effect by means of its discussions on various justifications for different types of statements. The learning unit, which the teachers planned and implemented, may have provided opportunities for confrontation with students' actual justifications and thus contribute to the teachers' repertoire of justifications. Finally, it is possible that the test given at the beginning of the course, which included suggestions for students’ justifications, influenced the teachers' knowledge.

References

Aigner, M., & Ziegler, G. (1998). *Proofs from THE BOOK*, Berlin: Springer.

Australian Education Council (1991). *A national statement on mathematics for Australian schools*. Melbourne: Curriculum Corporation.

Balacheff, N. (1991). The benefits and limits of social interaction: The case of mathematical proof. In A. J. Bishop, S. Mellin-Olsen, & J. van Dormolen (Eds.), *Mathematical knowledge: Its growth through teaching* (pp. 175-194). Dordrecht, The Netherlands: Kluwer Academic Publishers.

Bell, A. (1976). A study of pupils' proof-explanation in mathematical situations. *Educational Studies in Mathematics, 7*, 23-40.

Dreyfus, T. (2000). Some views on proofs by teachers and mathematicians. In A. Gagatsis (Ed.), *Proceedings of the 2nd Mediterranean conference on Mathematics Education*, Vol. I (pp. 11-25). Nikosia, Cyprus: The University of Cyprus.

Harel, G., & Sowder, L. (2007). Toward comprehensive perspectives on the learning and teaching of proof. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 805-842). Greenwich, CT: Information Age Publishing.

Healy, L., & Hoyles, C. (1998). *Justifying and Proving in School Mathematics*. London: Institute of Education, University of London.

Healy, L., & Hoyles, C. (2000). A study of proof conception in algebra. *Journal for Research in Mathematics Education, 31*, 396-428.

Hill, H. C., Ball, D. L., Sleep, L., & Lewis, J.M. (2007). Assessing teachers’ mathematical knowledge: What knowledge matters and what evidence counts? In F. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 111-155). Charlotte, NC: Information Age Publishing.

Israeli Ministry of Education (1994). *Tomorrow 98*. Jerusalem, Israel: Ministry of Education (in Hebrew).

Knuth, E. J. (2002). Teachers' conceptions of proof in the context of secondary school mathematics. *Journal of Mathematics Teacher Education, 5*, 61-88.

National Council of Teachers of Mathematics [NCTM] (2000). *Principles and standards for school mathematics*. Reston, VA: NCTM.

Stylianides, A. (2007). Proof and proving in school mathematics. *Journal for Research in Mathematics Education, 38*, 289-321.

Thurston, W. P. (1994). On proofs and progress in mathematics. *Bulletin of the American Mathematical Society 30*(2), 161-177.

Tsamir, P., Tirosh, D., Dreyfus, T., Barkai, R., & Tabach, M. (2008). Inservice teachers' judgment of proofs in ENT. In O. Figueras, J. L. Cortina, S. Alatorre, T. Rojano & A. Sépulveda (Eds.), *Proceedings of the 32nd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4 (pp. 345-352). Morélia, México: PME.

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