A FRAMEWORK FOR UNDERSTANDING THE STATUS OF EXAMPLES IN ESTABLISHING THE VALIDITY OF MATHEMATICAL STATEMENTS

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We offer a mathematical framework for dealing with the question "What does it mean to understand the status of examples in determining the validity of mathematical statements?" Our framework presents the interplay between four types of examples (confirming, non-confirming, contradicting, and irrelevant) with respect to the validity of two types of mathematical statements (universal and existential). This framework provides a basis for constructing tasks that assess and support students' understanding of the logical connections between examples and statements. We illustrate the strength of this framework by examining one student's evolving understanding of the status of examples in proving or refuting a mathematical claim, in the course of dealing with one of the tasks that were designed in accordance with the framework.

BACKGROUND

A tension between empirical and formal aspects of mathematics has been broadly recognised as one of the major sources of students' difficulties with proving (Epp, 2003; Fischbein, 1987). This tension can be seen in students' tendency to rely on specific examples as sufficient for determining that a general claim is true. Harel & Sowder (1998) term this type of reasoning *empirical proof scheme*, while Rissland (1991) and Zaslavsky & Shir (2005) refer to it as *example based reasoning*. At the same time, there is evidence that students have difficulties with counterexamples, thus often regard them as exceptions, insufficient to refute a (false) claim (Reid, 2002). This tension suggests that understanding of the logical relations between examples and statements is a non-trivial accomplishment, which is critical for proving. However, this kind of understanding is not usually addressed explicitly in school curriculum.

Our study stems from this unfortunate state of affairs and focuses on the question "What does it mean to understand the status of examples in determining the validity of mathematical statements, and how may this understanding be captured?" In order to answer this question be began with a mathematical analysis that partly builds on the works of Zaslavsky & Ron (1998), Buchbinder & Zaslavsky (2007), Barkai et al (2002). In particular, similar to the work of Barkai and her colleagues, we examine two types of mathematical statements (universal and existential). While they deal

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mainly with methods of proving, we specifically examine the status of examples in proving and disproving. We offer a general framework that can be used to guide investigation of this kind of mathematical understanding.

MATHEMATICAL FRAMEWORK

Generally, mathematical statements can be classified into two main categories: *universal* and *existential*, according to the type of quantifier that appears (sometimes implicitly) in the statement. A universal statement asserts that a proposition *P(x)* is true for all values of the variable *x* in a particular domain *D* ($\forall x \in D, P(x)$); while an existential statement asserts that there is an element in a domain *D* for which a proposition *P(x)* is true ($\exists x \in D, P(x)$).

We propose a framework and illustrate it through the domain *D - the set of all parallelograms*, and the proposition *P(x)* - indicating the property of *having two diagonals of equal length*. Thus, the corresponding universal statement is: *"All parallelograms have two diagonals of equal length"* while the corresponding existential statement is: *"There exists a parallelogram which has two diagonals of equal length"*. Depending on whether *x* is an element of the above domain *D* (that is, whether it is a parallelogram or not), and whether the proposition *P(x)* holds for it (that is, whether it has two diagonals of equal lengths or not), it can be classified as one of four types of examples (Figure 1).

Thus, we illustrate the first type of example by a rectangle, which is a parallelogram with equal-length diagonals ($x \in D, P(x)$); we term it a *confirming example*, for both the universal and existential statements. However, its status with respect to determining the truth value of these two types of statements is different: while, this example is sufficient for proving the existential statement, it is insufficient for proving the universal one.

The second type of example is an element of *D*, which does not satisfy the property *P(x)* ($x \in D, \neg P(x)$); for example, a (non-rectangle) rhombus, which is a parallelogram with unequal diagonals. With respect to the (false) universal statement it constitutes a counterexample that *contradicts* it, thus is sufficient for disproving it. However, with respect to the corresponding existential statement, this example violates the property *P*, does not confirm it, and yet does not contradict it either. Thus, the above rhombus is *non-applicable* for proving, and insufficient for disproving the existential statement. We term such example a *non-confirming example* for an existential statement, opposed to contradicting its corresponding universal statement.

The other two types of examples can be viewed as members of the same category since they both represent elements that do not belong to the domain *D*. As such, they are *irrelevant* for evaluating the validity of both universal and existential statements. Nonetheless, there is an important distinction between these two types of irrelevant

examples. An isosceles trapezoid is not a parallelogram[1] but it has two diagonals of equal length ($x \notin D, P(x)$), while a right-angle trapezoid, which in addition to not being a parallelogram has diagonals of unequal length ($x \notin D, \neg P(x)$), is another type of irrelevant example.

The Domain:	*D*: The set of all parallelograms			
The Proposition:	*P(x)*: Equal-length diagonals			
Type of Statement	Universal statement $\forall x \in D, P(x)$ (All parallelograms have equal-length diagonals)		Existential Statement $\exists x \in D, P(x)$ (There exists a parallelogram with equal-length diagonals)	
Goal / Type of Example	To prove	To disprove	To prove	To disprove
Confirming $x \in D, P(x)$  (a rectangle)	**Insufficient**	**Non applicable**	**Sufficient**	**Non applicable**
Contradicting (the universal statement) **Non-confirming** (the existential statement.) $x \in D, \neg P(x)$  (a rhombus)	**Non applicable**	**Sufficient**	**Non applicable**	**Insufficient**
Irrelevant $x \notin D, P(x)$  (an isosceles trapezoid)	**Non applicable**	**Non applicable**	**Non applicable**	**Non applicable**
Irrelevant $x \notin D, \neg P(x)$  (a right-angled trapezoid)	**Non applicable**	**Non applicable**	**Non applicable**	**Non applicable**

Figure 1: A framework for examining the status of examples in determining the validity of mathematical statements (illustrated with respect to a specific proposition)

[1] According to the definition accepted in many countries, a trapezoid has exactly two parallel sides.

As suggested by their name, irrelevant examples are non-applicable for neither proving nor disproving any type of mathematical statement. In spite of their logical resemblance, the two types of irrelevant examples may affect students' decision process quite differently.

We consider "understanding of the status of examples in determining the validity of mathematical statements" as becoming fluent with the logical inferences that can and cannot be made based on the different types of examples with respect to the validity of the two kinds of statements that appear in Figure 1.

THE STUDY

Goals

We report on one part of a larger study, the goals of which were:

1. To learn about students' existing conceptions associated with the roles and status of examples in determining the validity of mathematical statements;
2. To examine students' tendency to generate and use examples in the context of determining the validity of mathematical statements;
3. To trace the development of students' understanding of the roles and status of examples in proving and disproving.

Data Collection and Analysis

The data for this paper was collected in the course of an individual interview with Debby, a 10th grade, average level student, while she was dealing with one of the tasks, specially designed for the study (see below). No time constrains were set. She was also allowed to go back and check her previous answers. The interview (conducted by the first author) lasted for one hour. It was audio taped and transcribed.

Following our framework (Figure 1), the student's work was analysed in terms of the manifestations of her understanding of the status of examples in determining the validity of mathematical statements. In particular, four types of cognitive activity in which she was engaged were examined: example generation, example recognition (i.e., in terms of Figure 1), logical inference, justification and explanation supporting her inferences.

Instruments

The research instruments consist of a collection of tasks that were constructed according to our above framework. One of the tasks includes three false universal and three true existential statements, which deal with basic arithmetic and algebraic facts taken from the content of school mathematics (Figure 2). The task is to determine, for each statement, whether it is true or false, and to explain why. For most statements, several confirming, non-confirming and counterexamples can be found.

For each of the following statements determine whether it is true or false. Explain (prove) your answer.

1. If n is an even number, then $a^n > 0$ for any integer a.
2. If $(a+b)^2$ is an even number, then both a and b are even.
3. For any prime number, if we change the order of its digits, the resulting number is also a prime.
4. There exist two numbers a and b the sum of which is greater than one of them and smaller than the other.
5. There exists a two-digit integer for which the sum of its digits is equal to their product.
6. There exists a natural number n, the square of which is smaller than the number n itself.

Figure 2: The task used for the pilot study.

FINDINGS

Debby began with the first three (universal) statements and used both confirming and contradicting examples in her attempts to evaluate their validity. She was unpleasantly surprised by the fact that she had found these two types of examples that seemed to her that could not co-exist (she explicitly stated this later during the interview), nonetheless, she correctly concluded that these three statements were false. Debby claimed that she "…found an example that the statement is true and also an example that it is false…. Conclusion - the statement is false…it's not true in all cases".

Using our framework at this point to asses Debby's understanding of the logical connections between examples and statements, we could say that Debby was able to *construct* several relevant examples and *recognise* their type. She *inferred* and correctly *explained* that confirming examples were insufficient for proving the statements, but a single contradicting example was sufficient to disprove them. Inconsistencies in Debby's understanding only became apparent when she tried to apply the same techniques to evaluate existential statements.

For statement #4 (Figure 2) Debby found two non-confirming examples and one confirming example: $-1+1=0$. She concluded: "Here the statement is true. 0 is smaller than 1 and greater than (-1). This means that the statement can be true and [at the same time] can be false." When pressured by the interviewer to choose a single answer: "true" or "false", after an initial state of uncertainty, Debby decided that the statement is true: "It's true. […] I can't choose "false" if I give you a proof that it's true".

From Debby's correct answers to all the items in the task (up to this point), it seemed that she was fully aware of the distinction between universal and existential statements. However, her remark about the statement being "both true and false" stood in contrast with her actions, so we decided to ask her to further explain.

Int: What is the difference between statements #3 & #4?

Debby: The difference is ... that here it's true and here it's false. Here [# 3] we provided a proof that it is false and here [# 4] I gave you a proof that it is true.

This excerpt shows that Debby was either unaware that one statement is universal and the other is existential or that she just couldn't express it. Her response to statement #5 revealed that she was indeed unaware of the difference between universal and existential statements and of the status of confirming examples in determining the validity of the latter.

For statement #5 (Figure 2) Debby had found two non-confirming examples, and was just about to infer that there is no two-digit number for which the sum of its digits is equal to their product, when she discovered a confirming example.

Debby: I think no such number exists. Ah, sorry, it does. 22. The sum of the digits is 4, their product is also 4. So such statement exists. It means that the statement is….both true and false! So?! [...] I would choose "false". I can't determine that it's true when it's not true for all cases.

Note, that for both existential statements (#4 & #5) Debby had found two non-confirming and one confirming example, but inferred that statement #4 is true and statement #5 is false, because "it's not true for all cases". This inconsistency further indicated that Debby did not distinguish between universal and existential statements. In addition, she stated (repeatedly) that a statement can be true and false at the same time. However, Debby found this "conclusion" problematic, as indicated in the next excerpt:

"In all my previous answers it could be both true and false. And also here: it doesn't matter which answer I select! That's the problem. [...] For any answer I choose, even if it is true or false, I've *proved* that it is both true and false. So, any answer will be incorrect. What I could write is, that the statement is true but not for all cases. […] I see that it keeps coming up, so I think that something is wrong here. …I have to say this."

In order to help Debby resolve the uncertainty she was experiencing, we asked her to reflect again on her responses to statements #3, #4 & #5 and to try to reach consistency among them. After careful examination Debby replied:

Debby: Here [#3] they are talking about all numbers, and here [#4 & #5] they don't refer to all numbers… Here [#3] it can be both true and false, but they say "all numbers", so it has to be only one correct answer.

Int. : What do you mean?

Debby: A single answer. If I choose "false" it has to be "false". […] For these two [#4 & #5] they say that there are numbers that satisfy these statements. […] but they don't say that this is so in all cases.

Int. So is it true or false?

Debby: Both! Just like I said before. […] They say that there exist such numbers, but not all of them.

Int.: Have "they" asked for that?

Debby went back to compare her answers once again. After a while she said:

> I told you before that here [#4] it can be both true and false? I was wrong. It's only true.[...] I didn't understand the statement properly. I was sure that they are talking about all numbers, as I wrote here [#3]. Now I'm sure that #4 is true, and this answer [false] is out of the question. Beforehand, I thought that they mean all numbers, so I said that it can be both true and false. I understand now that they only say that such numbers exist. Not all of them, but there are such.

This was a moment of insight for Debby. After that she was able to determine and correctly justify that statement #5 is also true, and to complete the rest of the task.

We apply our framework to analyse Debby's initial and growing understanding of the logical connections between examples and statements in course of dealing with the task. All the examples that Debby *constructed* were relevant, and she was able to *recognize* their type (without using our terms): confirming, non-confirming or contradicting. For universal statements Debby was able to correctly *infer* their validity and logically *justify* her conclusion. The gaps in her understanding surfaced when she tried to use a rule of thumb "a theorem must be true in all cases" with existential statements (#4 & #5). Despite her ability to *construct* and *recognise* correctly the type of examples with respect to the statements, Debby had trouble to *infer* their truth value correctly or consistently (i.e., statements #4 & #5). She was especially confused by the existence of both confirming and non-confirming examples.

During the interview, we tried to minimize our intervention, allowing Debby to reason freely and reach the conclusions by herself. At the same time we kept insisting that she commits to a single response for each statement (true or false), and that she responds consistently. It seems that these two demands contributed to the development of Debby's understanding, by allowing her to become aware of the distinction between universal and existential statements, as well of the status of different types of examples in determining their validity. As she stated explicitly, this awareness helped her realise that a statement can be either true or false, but not both. Another indication of the development in Debby's understanding can be seen in her increasing ability to correctly *justify* her responses to the task.

CONCLUDING REMARKS

In this article we presented a framework for dealing with the status of examples in determining the validity of mathematical statements. As shown above, this framework proved useful in constructing tasks that elicit the logical connections between examples and statements, as well as in assessing this kind of understanding.

The specific features of the task, which included both universal and existential statements that have confirming and non-confirming or contradicting examples, seemed to contribute to the development of Debby's understanding. These features created for her a sense of uncertainty and doubt, which have been broadly recognized

as a powerful vehicle for learning mathematics (Hadas, Hershkowitz, & Schwarz, 2000; Zaslavsky, 2005). Debby's resolution of the state of uncertainty resulted in her better understanding of logical issues associated with the status of examples in determining the validity of mathematical statements.

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