A COMPARISON OF FOURTH AND SIXTH GRADE STUDENTS' REASONING IN SOLVING STRANDS OF OPEN-ENDED TASKS

Carolyn A. Maher — Rutgers University
Mary Mueller — Seton Hall University
Dina Yankelewitz — Rutgers University

This paper reports on the forms of reasoning elicited as fourth grade students in a suburban district and sixth grade students in an urban district worked on similar tasks involving reasoning with the use of Cuisenaire rods. Analysis of the two data sets shows similarities in the reasoning used by both groups of students on specific tasks, and the tendency of a particular task to elicit all forms of reasoning in both groups of students. Attributes of that task and ways that those attributes can be replicated in other domains may have implications in the teaching of early reasoning.

INTRODUCTION

The NCTM states in its *Principles and Standards for School Mathematics* (2000) that a primary goal of mathematics education in grades K-12 are the development of reasoning (and proof). Further, the document points out that students must be exposed to different forms of reasoning and must learn to choose and use appropriate forms of reasoning, citing: "Students need to encounter and build proficiency in all these [e.g., reasoning by contradiction, cases, and direct deductive reasoning] forms with increasing sophistication as they move through the curriculum" (p. 59).

In the study described in this paper, we examined the forms of reasoning that were elicited by two different groups of students, fourth graders and sixth graders, investigating problems from strands of open-ended tasks and providing justification for their solutions. We identified the occurrence of direct reasoning, reasoning by contradiction, reasoning using upper and lower bounds, reasoning by cases, and generic reasoning (as defined in Table 1 below). The research questions guiding our work are: (1) what similarities or differences in forms of reasoning are exhibited by both groups of students as they worked on equivalent tasks? (2) Do certain tasks tend to elicit particular forms of reasoning? And, (3) if so, what are their characteristics?

THEORETICAL FRAMEWORK AND RELATED LITERATURE

The ability to reason is crucial for students to develop both a need and appreciation for making convincing arguments. It is also a basic requirement for supporting arguments in justification and proof making in the learning of mathematics. Several studies have documented the ability of elementary and middle school students to reason and provide justification for their reasoning as they work collaboratively in a supportive environment (Ball, 1991; Lampert, 1990; Maher & Martino, 1996; Mueller, 2007; Mueller & Maher, 2008; Steencken, 2001; Steencken & Maher, 2003; Yackel & Cobb, 1994).

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Psychologists have defined reasoning as the process of coordinating evidence, ideas, and beliefs to draw conclusions about what is true (Leighton, 2003). Mathematical reasoning has been described by Yackel and Hanna (2003) as both the use of induction, deduction, association, and inference to draw conclusions about quantity and structure, and a "communal activity in which learners participate as they interact with one another to solve mathematical problems" (p. 228). The second description situates mathematical reasoning in a context that allows for it to be elicited, while the first qualifies the forms of reasoning that are useful when doing mathematics. This study builds on and extends the research by examining patterns in the forms of reasoning elicited by elementary and middle school students as they worked on open ended tasks.

METHODOLOGY

This study draws from two data sets. The first is a longitudinal study of students' mathematical thinking that was conducted by researchers in a fourth grade classroom of twenty-five students in a suburban school in New Jersey[1]. The second source of data is an informal after-school math program consisting of twenty-four sixth grade students that was conducted by researchers in an low socioeconomic, urban community in New Jersey, drawn from a school consisting of 99% Latino and African American students[2]. In the schools of this study, computation with fractions is introduced in grade 5. Hence, the fourth graders were not yet taught operations with fractions and related fraction ideas, while those in grade six had been taught procedures related to fraction ideas and operations. For both classes, the series of sessions were videotaped with at least two cameras. For this paper, we report data from the first seven 60 minute sessions from the fourth grade study and the first five 60-75 minute sessions from the sixth grade study[3]. The fourth graders began their work in pairs; while the sixth graders worked in groups of three or four. Both groups investigated tasks involving Cuisenaire rods. The strands of tasks were the same or very similar in both studies. Students were encouraged to provide justification for their solutions and to challenge and question the explanations of others.

Video recordings and transcripts were analyzed using the analytical model outlined by Powell, Francisco & Maher (2003). The transcripts were coded for forms of reasoning. For the purposes of this study, the forms of reasoning were defined as

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[3] See Mueller 2007 and Mueller & Maher, 2008 for a detailed analysis of the second set of data

follows. First, *direct reasoning* was based on the assumption that "the hypothesis contains enough information to allow the construction of a series of logically connected steps leading to the conclusion" (Cupillari, 2005, p. 12). Second, *reasoning by contradiction* was based on the agreement that whenever a statement is true, its contrapositive is also true; or that a statement is equivalent to its contrapositive (Cupillari, 2005). Third, *reasoning by cases* involved students defending an argument by defending separate instances. Fourth, *reasoning using upper and lower bounds* was noted when the upper and/or lower bound of a subset S of some partially ordered set was defined, and an argument was then formed to justify a statement about the subset within the defined bounds (for example, that it is empty). Fifth, *generic reasoning* involved reasoning about a paradigmatic example whose properties can be applied to the set under discussion and lends insight into a more general truth, which in turn verifies the claim made about the particular example (Rowland, 2002)

Finally, the two data sets were compared, and similarities and differences were noted.

Task	Grade 4	Grade 6
If we call the dark green rod one, what number name would I give the light green rod?	Direct	Direct
Someone told me that the red rod is half as long as the yellow rod. What do you think?	Contradiction	Contradiction
If we call the blue rod one, which rod will have the number name one half?	All 5 forms noted	All 5 forms noted
If we call the orange rod one, what number name will I give the white rod?	Direct	Direct
If we call the orange rod one, what number name will I give the red rod?	Direct	Direct
Is 1/5 = 2/10?	Direct	Direct & Contradiction

Table 1: Sample Tasks and Forms of Reasoning that Emerged

RESULTS

Throughout the sessions, all students contributed to the discussion and direct and indirect forms of reasoning were elicited. Analysis of the data showed similarities between the forms of reasoning used by students in both grades. A representative sample of tasks that elicited similar forms of reasoning in both groups of students is listed in Table 1. Due to space limitations, this paper will discuss the similarities between the reasoning used by the fourth and sixth graders as they worked on the task: "If we call the blue rod one, which rod will have the number name one half?"

This task was posed during the second session of each study, and both groups of students worked to find the nonexistent rod that could be called one half of the blue rod (which is nine centimeters long). In both groups of students, the arguments were rich and varied, and correct lines of reasoning by contradiction, upper and lower bounds, cases, and generic reasoning were presented. In both groups, the students used direct reasoning to offer faulty solutions to the task.

The Fourth Grade

Reasoning Using Upper and Lower Bounds

In the fourth grade, David offered a solution and presented an argument using upper and lower bounds (c.f. Maher & Davis, 1995). He said, "I don't think that you can do that because if you put two yellows that'd be too big, but then if you put two purples … that'd be too short". When questioned by the researcher if there was any rod between the purple and the yellow, David replied, "I don't think there is anything."

David then presented his argument at the overhead projector. He showed that the length of the purple rod was one white rod shorter than the yellow rod, and then lined up the rods in a "staircase" pattern to show that, when ordered according to length, each successive rod was one white rod longer than the previous rod. He demonstrated that there is no rod that is shorter than the yellow rod and also longer than the purple rod.

Direct Reasoning (incorrect)

Erik incorrectly reasoned that the purple and yellow rod could each be called one half of the blue rod. Erik said, "See, I figured if you take a yellow and a purple it's equal [to the length of the blue rod]. They're not exactly the same, but they're both halves. Because the purple would be half of this even though the yellow is bigger because if you put the purple on the bottom and the yellow on top it's equal, so they're both halves, but only one's bigger than the other. So it equals up to the same thing."

Reasoning by Contradiction

Many students disagreed with Erik's suggestion. Alan and Jessica used the definition of one half to counter Erik's claim.

Alan: When you're dividing things into halves, both halves have to be equal – in order to be considered a half.

Jessica: This isn't a half. Those two aren't both even halves.

Alan made the definition of *one half* explicit. Jessica then showed the contradiction inherent in Erik's argument by saying that the purple and yellow rods aren't the same length, and therefore cannot be called one half of the blue rod. This counterargument is a sophisticated use of reasoning by contradiction, as it considers the faulty claim that Erik proposed and uses a definition to show the contradiction in the argument.

Reasoning by Cases

David then used an argument by cases to show that all rods can be classified as even or odd, and that their ability to be divided in half can be determined from that classification. He showed that the white, light green, yellow, black, and blue rods are "odd" because no rod exists that is one half their length. He then showed that the remaining rods are "even". To illustrate his point, he showed that two purple rods equal the length of the brown rod and two yellow rods are equal to the length of the orange rod, proving that the brown and yellow rods are even.

Generic Reasoning

David's argument by cases also contained an element of generic reasoning. By using the example of the blue rod, he showed that its properties can be applied to all "odd" rods in the set. After showing this general truth, he returned to the specific case of the blue rod, showing that since it belongs to the category of "odd" rods, there does not exist a rod that is half its length.

The Sixth Grade

In the sixth grade informal math session, similar arguments were used. There, direct reasoning, generic reasoning, and reasoning by cases, upper and lower bounds, and contradiction were used, and their arguments closely resembled those of the fourth grade students.

Direct Reasoning (incorrect)

Michael and Shirelle each proposed a solution similar to Erik's, suggesting that the yellow and purple rods are two halves of the blue rod. They showed that a purple and yellow train equals the blue rod, and called each of the rods half of the blue rod.

Reasoning by Contradiction

Chris used an argument by contradiction to show that there was no rod whose length was half of the blue rod. Building a model of nine white rods lined up next to the blue rod, he said, "Since there's nine white little rods you can't really divide that into a half so you can't really divide by two because you get a decimal or remainder so there is really no half, no half of blue because of the white rods."

Chris argued that since there were nine of the smallest size rods, and nine is not divisible by two, there was no rod that was one half of the blue rod. He used the fact that each rod was equal in length to a multiple of the white rods, and that if the length of the blue rod couldn't be divided into two equal groups of white rods, no rod that was half of the blue rod existed. Chris' use of decimals and remainders in his argument highlights one difference between the two groups of students. Due to their more extensive exposure to mathematics, the sixth graders included some mathematical ideas in their arguments that were not used by the fourth graders.

Reasoning Using Upper and Lower Bounds

Dante presented an argument using upper and lower bounds at the overhead projector. After repeating his argument by contradiction that the yellow is longer than the purple by one white rod and so each cannot be one half of the blue rod, he showed that there are no other rods that can be called one half of the blue rod. He said, "Usually for the blue piece, it would usually be purple or yellow but yellow would be one um one white piece over it and the pink would be, I mean purple would be one white piece under it."

Dante argued that either the purple or the yellow would be possible halves of the blue rod. However, he explained that the length of two yellow rods is one white rod longer than the blue rod, and that the length two purple rods is one white rod shorter than the blue rod. This argument showed that the yellow rod is an upper bound and the purple rod is a lower bound, and that there is no rod that is exactly one half of the blue rod.

Reasoning by Cases, Generic Reasoning

Justina then presented an argument by cases similar to David's. She called David's "odd" rods "singles", since those rods cannot be paired with a rod that is half its length. She then showed that the white, light green, yellow, black, and blue rods fall into this category. Similar to David's argument, her argument also contained an element of generic reasoning.

CONCLUSIONS AND IMPLICATIONS

In both studies, all forms of reasoning identified prior to the study were elicited, and varied arguments were presented to justify solutions that were offered. Many similarities in forms of reasoning used were noted, as can be seen from the sample results in Table 1. It can be concluded that careful task design can enable all students to reason effectively and learn to use different forms of arguments as they do mathematics. One significant difference that was noted between the forms of reasoning elicited for the different tasks. For the majority of tasks posed during the sessions in both studies, direct reasoning was the most common form used by the students. For example, of the six tasks analyzed in the table, four of the tasks elicited direct reasoning. When working on the task presented above, however, direct reasoning was only used to make incorrect claims, while reasoning by cases, contradiction, upper and lower bounds, and generic reasoning was used to provide correct solutions to the problem. This discrepancy highlights the tendency of this task to elicit forms of reasoning other than direct reasoning and the fact that different tasks tend to elicit different forms of reasoning consistently.

The discrepancy that was noted in the results pointed to a possible answer to the third and fourth research questions. We examined the task that elicited these different forms of reasoning to identify the characteristics of the task that may have been the cause of this difference. One possible reason for this difference is the structure of the task. Many other tasks required students to show the truth of a statement, and lent themselves to direct arguments. This task, however, encouraged students to show an argument by contradiction since, in the given set, there was no rod with number name one half, when blue was called one. This invited students to use other forms of reasoning, as called for in the NCTM standards, and resulted in the display of rich, varied arguments. The significant differences between the two groups under study, including their differences in age and socioeconomic status, as well as the different settings in which the tasks were posed, suggest that carefully designed tasks can provide the opportunity for all students to gain knowledge in using different forms of reasoning, meeting a primary goal of mathematics education. In order for students to practice using mathematical reasoning in all its forms, it might be helpful for teachers to introduce tasks that have the potential to elicit various forms of reasoning.

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