PUPILS' EXPLAINING PROCESS WITH MANIPULATIVE OBJECTS

Kotaro Komatsu

Graduate School of Comprehensive Human Sciences, University of Tsukuba, Japan

The purpose of this study is to analyze how pupils can explain why a statement is true by utilizing manipulative objects. A pair of 5th graders was asked to conjecture statements on properties of some two-digit additions and explain them with coins. As a result, they were able to invent explanations of why the statement was true because they intentionally continued to coordinate their enactive and symbolic representations. Besides, the mathematical structure of coins facilitated pupils' coordination of the representations. Pupils' activity is also discussed in terms of mathematical understanding, and some important implications for teaching are mentioned.

INTRODUCTION

Proof and proving are at the heart of mathematics and should therefore be learned from elementary school mathematics (Stylianides, 2007). Though the main function of proof in mathematics education is to explain why a statement is true (Hanna, 1995), some researches show that it is difficult for pupils to explain it. According to Monoyiou, Xistouri & Philippou (2006), few 5th and 6th graders (only 3.4 %) could describe the general reason of "the sum of two odd numbers is even". Stylianides (2007) also reports a class where a 3rd grader explained why the above statement was true, but some pupils were not convinced by her explanation, and objected to her explanation.

However, it may be that pupils can express deductive reasoning better if they are allowed to utilize manipulative objects. Based on Piaget's researches (cf. Piaget, 1953), Semadeni (1984) points out that it is difficult for concrete-operational children to reason deductively using only words and symbols. And as primary-school proofs, he proposes "action proofs", which consist of performing "certain concrete, physical actions (manipulating objects, drawing pictures, moving the body etc.)" (Semadeni, 1984, p. 32). In addition, Miyazaki (1992) shows the process by which a 6th grader proves the generality of his conjecture with manipulative objects. Perhaps, pupils would be also able to explain more easily why a statement is true utilizing manipulative objects. Therefore, the focus of this study is to analyze how pupils can explain why a statement is true by utilizing manipulative objects.

THEORETICAL FRAMEWORK

Though there are many functions of proof in mathematics, their function in mathematics education is primarily to explain why a statement is true. De Villiers (1990) points out five functions of proof in mathematics: verification, explanation, systematisation, discovery, and communication. According to Hanna (1995),

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, pp. 393-400. Thessaloniki, Greece: PME.

explanation is very important function in mathematics education, because we can deepen mathematical understanding by explanation. It is therefore valuable for pupils to learn to explain why a statement is true. Moreover, as Mancosu (2001) shows, the nature of explanation continues to be a "hot topic" in the philosophy of mathematics.

Pupils will feel the need for explaining if they conjecture statements by themselves and feel some surprise or doubt from the conjecture. For example, Hadas & Hershkowitz (1998) conclude, from students' activities in which dynamic geometry software was used, that the need for explanation was heightened by "a surprise caused by the contradiction between the conjectures and what students got (or could not get) while working the dynamic tool" (p. 32). Hence, the design of tasks which cause "surprise" is important for motivating pupils' explaining.

This explanation can be represented by not only formal proofs (written in mathematical language or symbols) but also informal proofs (in which one also uses manipulative objects or diagrams). Hanna (1990) distinguishes between "proofs that explain" and "proofs that prove", and shows a geometric proof of the sum of consecutive natural numbers as an example of the former. Miyazaki (1995) also argues that it is possible to explain by proofs based on concrete actions, and discusses what students should do in order to do that. Further, according to Piaget (1953) or Semadeni (1984), concrete-operational children are able to perform deductive reasoning better with manipulative objects. Therefore, if pupils are allowed to access manipulative objects, it should become easier for them to explain why a statement is true.

When one utilizes manipulative objects, thinking explanation through relating enactive and symbolic representations becomes important in terms of deepening mathematical understanding. Hiebert & Carpenter (1992) state that a "mathematical idea, procedure or fact is understood ... if its mental representation is part of a network of representations" (p. 67). They argue that one way of constructing such a network is by making connections between different representation forms of the same mathematical idea. However, relating enactive and symbolic representations in proving has not been examined so far. For example, in Miyazaki (1992, 1995), pupils seek proofs only in enactive representation. Miyazaki (2000), who deals with transition from proofs based on concrete actions to algebraic proofs, also discusses one-way relation, that is, translation of enactive into symbolic representations. It is therefore valuable and necessary to analyze pupils' explaining process from the viewpoint of whether and how they relate enactive and symbolic representations.

METHOD

A pair of 5th graders, Hina and Yui (pseudonyms), who belong the same class in a public elementary school was interviewed. The reason why the pair was chosen is that, according to their teacher, they can communicate with each other actively. Further, in the interview they had access to coins, papers and to only one pencil, because Balacheff (1988) suggests that pupils are likely to express their thinking more naturally in such environment. Their teacher said that their grades in Mathematics are Hina (excellent) and Yui (average), and that they have little experience with explanation. He also told that they do not often utilize concrete materials in their learning of mathematics; for example, they rarely use manipulative objects in considering methods of calculations and sometime use papers in learning geometrical figures.

First, they were asked to calculate “the sum of a two-digit number and the number where the original number digits are reversed” (for example, 32+23=55) in some cases, and conjecture statements from the results. Next, they were given coins (Japanese 100-yen, 10-yen and 1-yen coins), and asked if they could explain why the statements were true by utilizing the coins. The interviewer (the author) intended to intervene only if the two pupils seemed to reach a deadlock or if their thinking was ambiguous. The interview was recorded, and the transcripts were analyzed through the video recode.

RESULTS

On the above two-digit additions, Hina and Yui conjectured two statements. The first one (statement A) was that “(when the sum remains a two-digit number) the tens digit is equal with the ones digit [1)]” (for example, 32+23=55). The second one (statement B) was that "when carrying occurs, the sum of the hundreds digit and the ones digit is equal to the tens digit” (for example, 97+79=176, 1+6=7). Because the two pupils felt more “surprise” at statement B, their enactive and symbolic explanations of why statement B is true will be the focus of what follows.

Grasping the Statement with Manipulative Objects

Firstly, Hina and Yui placed 10-yen and 1-yen coins as Procedure 1 of Figure 1 in order to represent 39+93. At that time, they placed 10-yen coins and 1-yen coins on the left row and the right row respectively with consciousness of forms of “calculations with figures”. For, when they previously examined the reason why statement A was true, Hina said “we did in the case of calculations with figures, so we tried like that”, and they arranged coins on 23+32 in the same form as Proc. 1 of Fig. 1.

Next, Hina said “after all, it was same as before” and confirmed that the number of 10-yen coins was equal with the number of 1-yen coins, referring to the case of statement A and the commutative law of addition. She then stated that carrying occurs when there were more than ten coins, and they rearranged coins as Proc. 2 of Fig. 1. After that, Hina said “oh, the opposite, this, this comes to hundreds” and moved a set of 10-yen coins from the center row to the left row (from Proc. 2 to Proc. 3 of Fig. 1). They would represent this set as a hundred (but actually there were only nine 10-yen coins in this set). Thus, when they moved the set at this time, they also seemed to be conscious of “positional notation”.

Moreover, they exchanged ten 10-yen coins for a 100-yen coin (Proc. 4 of Fig. 1). They then regarded a set of ten 1-yen coins as equivalent to a 10-yen coin, and verified that the sum of this set and two 10-yen coins was equal with the sum of a 100-yen coin and two 1-yen coins (both were three). However, as they said “why..., um...”, they felt embarrassed and could not find the reason why the two sums became equal.

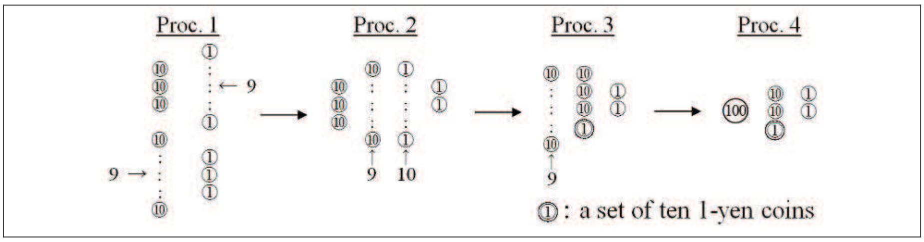

Fig. 1: The process in which the pupils arranged coins on 39+93

Explaining why the Statement is True

After Hina and Yui arrived at the arrangement shown in Proc. 4 of Fig. 1, they broke the arrangement and similarly rearranged coins on 48+84 as Fig. 2. Hina then referred to the arrangement and calculated 48+84 with figures, through explicitly separating the tens and ones places (the left-hand side in Fig. 3). According to their teacher, they do not usually write such forms in calculations with figures. This novel approach is essential to their explanation.

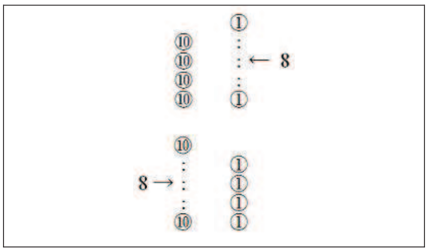

Fig. 2: Pupils' arrangement of coins

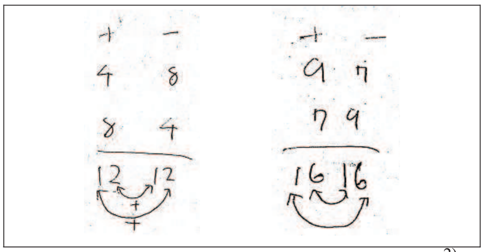

Fig. 3: Pupils' calculations with figures [2)]

Next, they added arrows to the calculation like the left-hand side in Fig. 3, and found that the sum of 2 of 12 in the tens place and 1 of 12 in the ones place was equal with the sum of 1 of 12 in the tens place and 2 of 12 in the ones place. After that, Hina proposed to consider other examples, and they verified that they could apply the same thinking to the case of 97+79 (the right-hand side in Fig. 3). Further, they regarded 97+79 as "a generic example" (Balacheff, 1988) of all two-digit additions, and described their explanation of the general reason why the statement B was true (Fig. 4).

It is clear that pupils' novel reconstruction of the form of the calculation with figures contributed to their invention of such explanation. Moreover, they could produce this novel symbolic reconstruction because they have consciously related positional notation or their calculations with figures (symbolic representation) and their arrangement of the coins (enactive representation). For example, their attention to calculations with figures led them to place 10-yen coins and 1-yen coins on the left row and the right row respectively in Fig. 2. Then, referring to the arrangement of coins, they calculated 48+84 by explicitly separating those in the tens and ones places (Fig. 3). Furthermore, their symbolic reconstruction is precisely matched by their insightful re-positioning of the coins shown in the move from Proc. 1 to Proc. 4 in Fig. 1 although they did not mention this correspondence explicitly.

くり上がった場合，十の位は，一の位のくり上がった数
と十の位のくり上がってない数をたしたもので，
百の位と一の位をたしたものと順番だけいれかえた
ものなので，答えは同じになるから。

When carrying occurs, the tens digit is the sum of one carried from the ones place and the number left in the tens place. Because the sum is reverse to the sum of the hundreds digit and the ones digit [3], the two sums become equal.

Fig. 4: Pupils' explanation why the statement B is true [4]

Explaining why the Statement is True with Manipulative Objects

After the two pupils described their explanation in Fig. 4, the interviewer asked them, "How about using coins?". They placed coins on 84+48 as Proc. 1 of Fig. 5, and told that both the numbers of 10-yen and 1-yen coins were twelve. They then transformed the arrangement from Proc. 1 to Proc. 5 of Fig. 5, and stated that the sum of 10-yen coins left (two) and a set of ten 1-yen coins represented the tens digit. Moreover, as the reason why the sum was equal with the sum of a set of ten 10-yen coins and 1-yen coins left (two), they mentioned three facts; both the numbers of 10-yen and 1-yen coins left were equal (two); both the numbers of a set of ten 10-yen coins and a set of ten 1-yen coins were equal (one); and they could therefore apply the commutative law of addition. At this point, they regarded a set of ten 10-yen coins and a set of ten 1-yen coins as a 100-yen coin and a 10-yen coin respectively, and explained why the sum of the numbers of 100-yen and 1-yen coins was equal with the number of 10-yen coins. Further, the three facts matched with the previous symbolic explanation.

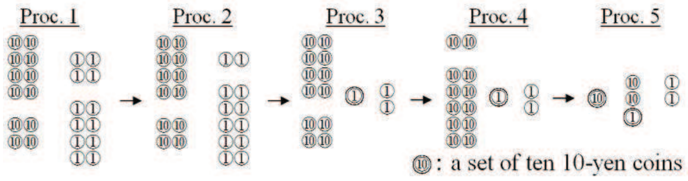

Fig. 5: The process in which the pupils arranged coins on 84+48

Then, the interviewer asked them whether they could apply the same thinking to all other cases and, if so, why they could apply. Hina answered, referring to the calculation of 97+79 in Fig. 3, "in other cases, only the order is reversed too, so I think the same thinking can be applied". Her answer seemed to mean that the above three facts hold true in all cases. Therefore, when Hina considered the arrangement of coins on 48+84 (Fig. 5), she would regard the coins as a generic example of coins which represented all two-digit additions, and show the general reason why the sum of the numbers of 100-yen and 1-yen coins became equal with the number of 10-yen coins, with awareness of the previous symbolic explanation.

DISCUSSION

Unlike other studies on pupils' explanation with manipulative objects (Miyazaki, 1992, 1995, 2000), Hina and Yui could explain why the statement B was true by shifting from an enactive representation to a symbolic representation, and vice versa, many times. For example, they grasped the statement B, which they conjectured in symbolic representation, as "the sum of the numbers of 100-yen and 1-yen coins is equal with the number of 10-yen coins" in enactive representation. At that point, they arranged coins with consciousness of calculations with figures and positional notation in symbolic representation. On the other hand, referring to the arrangement of coins, they explicitly separated the tens and ones places in the calculations with figures, and could therefore invent symbolic explanation of why the statement B was true. Moreover, when they produced enactive explanation with coins, the enactive explanation corresponded to the previous symbolic explanation (for instance, the sum of 10-yen coins left and a set of ten 1-yen coins represented tens digit.)

One of the most important reasons for the structure and fluency of their explanations was that they continued to coordinate their enactive and symbolic representations quite explicitly. As Hiebert & Carpenter (1992) discuss, relating various representations leads to deepen mathematical understanding. However, Duval (2006), who discusses mathematical comprehension in terms of registers of semiotic representations, points out that if there is no structural correspondence between two representations, translation "is for many students an impassable barrier in their mathematics comprehension and therefore for their learning" (p. 123). The two pupils in this study transformed the arrangement of coins in order to match it with positional notation, and, in turn, invented novel notations in calculations with figures to match the arrangement of coins. In this way, they intentionally continued to coordinate two representations through transforming the content of one representation in order to match it with that of the other representation. Because of their coordination, they would be able to explain why the statement B was true and deepen their understanding.

Another reason is that the mathematical structure of the coins used (the ratio of a 10-yen and a 1-yen coin is 10 : 1) matches the structure of the two-digit numbers. In this study, Hina and Yui conjectured statements on digits in the decimal notation system. Japanese 100-yen, 10-yen and 1-yen coins can be used to match the decimal notation system, *and* the numbers of coins correspond to digits in the system. It seemed that this structural correspondence had strong effect on pupils' coordination of enactive and symbolic representations. Conversely, if 50-yen and 5-yen coins had been utilized as well, or if 10-yen coins had been not available, such environments might have been obstacles for pupils' coordinating and explaining. Therefore, such promising results appear to be possible only if the structure of the manipulative objects is able to match the structure of the mathematical activities that pupils are engaged in.

From the result of this study, we can get two implications for teaching. Firstly, when pupils try to explain with manipulative objects why a statement is true, teachers should look for ways to encourage them to coordinate enactive and symbolic representations.

This coordination would promote pupils' explaining and deepen their mathematical understanding. Secondly, if possible, teachers should prepare manipulative objects whose mathematical structure matches with expected pupils activities. Such manipulative objects could encourage pupils' coordinating and explaining.

CONCLUDING REMARKS

Starting with an enactive model which they were able to reconstruct verbally and symbolically, a pair of 5^{th} graders in this study could explain symbolically why a mathematical statement was true. Subsequently, they were able to translate their symbolic explanation by reference to variants of their enactive models. This coordination was valuable in terms of mathematical understanding, and was facilitated by the mathematical structure of manipulative objects. These findings were influenced by the particular subject matter and the choice of manipulative objects. It is therefore necessary to examine these findings more thoroughly by interviewing other pupils and/or using other subject matters and manipulative objects.

Acknowledgements

The author would like to express his special thanks to Mr Yasuo Yamaguchi and his two pupils for their cooperation. He is also grateful to Dr Max Stephens, Dr Andreas Stylianides and Dr Yoshinori Shimizu for their valuable suggestions on earlier drafts of this paper.

Footnotes

1) Parenthesis by the author. Pupils' Japanese was translated to English by him.

2) "+" and "-" at the top of the calculation are Japanese characters. "+" means the tens place and "-" means the ones place, respectively.

3) If this clause is literally interpreted, the orders of the two sums are not reverse but the same. Though the two pupils intended to describe the reason why the commutative law of addition could be applied, they seemed not to care the orders in writing this clause.

4) Before writing this sentence, the two pupils stated that if the tens and ones place were calculated separately, they could get the same two two-digit numbers because of the commutative law of addition, for example 9+7=7+9=16.

References

Balacheff, N. (1988). A study of students' proving processes at the junior high school level. In Wirszup, I. & Streit, R. (eds.). *Developments in School Mathematics Education around the World: Applications-Oriented Curricula and Technology-Supported Learning for all Students, Vol. 2* (pp. 284-298). Reston, VA: NCTM.

De Villiers, M. (1990). The role and function of proof in mathematics. *Pythagoras, 24*, 17-24.

Duval, R. (2006). A cognitive analysis of problems of comprehension in a learning of mathematics. *Educational Studies in Mathematics, 61*(1-2), 103-131.

Hadas, N. & Hershkowitz, R. (1998). Proof in geometry as an explanatory and convincing tool. In Olivier, A. & Newstead, K. (eds.). *Proceedings of the 22nd Conference of the IGPME, Vol. 3* (pp. 25-32). University of Stellenbosch.

Hanna, G. (1990). Some pedagogical aspects of proof. *Interchange, 21*(1), 6-13.

Hanna, G. (1995). Challenges to the importance of proof. *For the Learning of Mathematics, 15*(3), 42-49.

Hiebert, J. & Carpenter, T. P. (1992). Learning and teaching with understanding. In Grouws, D. A. (ed.). *Handbook of Research on Mathematics Teaching and Learning: A Project of the National Council of Teachers of Mathematics* (pp. 65-97). New York: Macmillan Publishing Company.

Mancosu, P. (2001). Mathematical explanation: Problems and prospects. *Topoi, 20*, 97-117.

Miyazaki, M. (1992). Students' activity in order to show the generality of a conjecture: How does one student use a generic example to make an explanation?. *Journal of Japan Society of Mathematical Education, Reports of Mathematical Education, 57*, 1-19. (in Japanese)

Miyazaki, M. (1995). A study on showing the reason of universal validity of a proposition by acting with material objects. *Journal of Science Education in Japan, 19*(1), 1-11. (in Japanese)

Miyazaki, M. (2000). Levels of proof in lower secondary school mathematics: As steps from an inductive proof to an algebraic demonstration. *Educational Studies in Mathematics, 41*(1), 47-68.

Monoyiou, A., Xistouri, X. & Philippou, G. (2006). Primary students' reasoning in problem solving and teachers' evaluation of their arguments. In Novotná, J., Moraová, H., Krátká, M. & Stehlíková, N. (eds.). *Proceedings of the 30th Conference of the IGPME, Vol. 4* (pp. 177-184). Charles University in Prague.

Piaget, J. (1953). *Logic and Psychology*. Manchester: Manchester University Press.

Semadeni, Z. (1984). Action proofs in primary mathematics teaching and in teacher training. *For the Learning of Mathematics, 4*(1), 32-34.

Stylianides, A. J. (2007). The notion of proof in the context of elementary school mathematics. *Educational Studies in Mathematics, 65*(1), 1-20.