THE ROLE OF THE TEACHER IN DEVELOPING PROOF ACTIVITIES BY MEANS OF ALGEBRAIC LANGUAGE

Annalisa Cusi and Nicolina A. Malara

Dipartimento di Matematica, Università di Modena e Reggio Emilia

This work is part of a wider project of didactical innovation aimed at fostering students' conscious use of algebraic language through teaching experiments on proof in elementary number theory (ENT). In this paper we will analyze the role played by the teacher as a guide to the enactment of fundamental skills for the construction of reasoning by means of algebraic language, pointing out the specific characteristics of the teacher's effective action in serving as a role model to his/her students.

THEORETICAL FRAMEWORK

The work presented herein is part of a wide study (Cusi 2009a), within the framework of the Italian research for innovation (Arzarello & Bartolini Bussi 1998). It was conceived to promote, in the actual school environment, activities aimed at helping students in developing symbol sense (Arcavi 1994) towards a conscious use of algebraic language as a thinking tool (Arzarello *et Al.* 2001), possibly re-converting their previous pseudo-structural conception of algebra (Sfard & Linchevski 1994). This led to designing and implementing an innovative introductory path to proof in ENT (grades 9-10), in the frame of an approach to the teaching and learning of algebra focussed on the control of meanings (Cusi 2009b).

Our research takes place in a Vygotskyan perspective of approach to the teaching and learning processes, specifically with reference to: (a) the central role played by social interaction in thought-development processes (Vygotsky, 1978); (b) the importance of experts' contribution in helping students bridging the gap between their potential and their actual mental development (Vygotsky, 1978); (c) the role played by activities performed in a social context in the development of both the personal meaning that an individual attributes to them and the motives that determine his/her awareness of learning processes (Leont'ev, 1977). In this framework, the role played by the teacher is fundamental. This role (has been and) is the topic of several studies, especially as far as collective discussions are concerned. If we look at the social aspects of interaction, it turns out that the teacher should be able to create a good context for classroom discussions, by stimulating and regulating argumentative processes (Schwarz et Al., 2004) and, most of all, focusing students' attention onto the need to listen to others carefully, so that they might decide whether what they say makes sense and, possibly, criticise or give suggestions (Wood 1999). If we look at the mathematical content of the discussion, the teacher is in charge of determining the direction this content should take in the different phases of the discussion, "filtering" students' ideas, so that they can focus on those contents the teacher considers more relevant and meaningful (Sherin, 2002). Besides this, the teacher is in charge of the quality of the discussion because it is through his/her reactions to students'

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 2, pp. 361-368. Thessaloniki, Greece: PME.

interventions that he/she implicitly evaluates the solutions they propose (Yackel & Cobb, 1996), leading them to become aware of the finest forms of reasoning (Anghileri, 2006). Many researches have highlighted methodologies and behaviours to approach mathematical discussion, that the teacher should be able to manage and develop in a flexible and dynamic way (Anghileri 2006; Leikin & Dinur 2007).

The role played by the teacher becomes even more complex and delicate when mathematical proof is the object of discussion. Balacheff (1991) underlines how, sometimes, interaction may be an obstacle to learning within proving processes, because it favours the development of those "argumentative behaviours" which contrast with the achievement of an awareness of both the specificity of mathematical proof supporting one's conjectures and the role played by logical deduction. More recently, Martin *et Al*. (2005) have drawn attention to the importance of whole-class discussions and the role they play in the learning of mathematical proof, highlighting the impact of teachers' choices and actions both on individual and collective understanding. The authors stress that the teacher is in charge of favouring the inclusion of students in the process of class-based negotiation of a conjecture, as well as in the process of collective construction of the proving reasoning referring to precise rules. Moreover, Blanton *et Al.* (2003), exploring the role of instructional scaffolding in the development of undergraduate students' understanding of mathematical proof, highlight that students who engage in whole-class discussions that include metacognitive acts make gains in their ability to construct proof. These aspects also concern the construction of proofs by means of algebraic language, since it requires that students employ skills that may be attained only when they participate in contexts, such as those advocated by Schoenfeld (1992), wherein mathematical sense-making is practiced and developed. In previous works (see, for instance, Cusi 2009b) we highlighted those skills that students must develop in order to become able to succeed in this kind of activity and to acquire an awareness of the role played by algebraic language within these processes. In order to analyse the development of thought processes carried out by means of algebraic language we referred to some theoretical constructs, in tune with the view of teaching algebra that we are promoting, drawn from the works of Arzarello *et Al*. (2001), Boero (2001) and Duval (2006). The first authors highlight the use of conceptual frames (defined as an "organized set of notions, which suggests how to reason, manipulate formulas, anticipate results while coping with a problem") and changes from a frame to another as fundamental steps in the activation of the interpretative processes of the expressions progressively constructed. Boero focus on the concept of *anticipating thought* as a key-element in producing thought through processes of transformation (he defines anticipating as "imagining the consequences of some choices operated on algebraic expressions and/or on the variables, and/or through the formalization process"). Duval, who defines representation registers as those semiotic systems "that permit a transformation of representations", asserts that a critical aspect in the development of learning in mathematics is represented by changes of representation, be they either from one *representation register* to another (conversions) or within a single register (treatments). The analysis of students' discussions enabled us to highlight the role played, in the development of proofs in ENT, by three essential components (a. the appropriate activation of conceptual frames and coordination between different frames; b. the correct application of appropriate anticipating thoughts; c. a good flexibility in the coordination between algebraic and verbal registers) and the mutual relationships existing between them (Cusi 2009b). Difficulties met in making students develop the essential skills we have outlined, as well as in helping them become aware of the meanings that algebraic language can transmit, if used appropriately, enabled us to highlight the crucial role played by the teacher as a model students should refer to. This led us to analyze teacher's actions during those class discussions aimed at introducing the construction of proofs in ENT. This paper propose the main results of this part of our research.

RESEARCH HYPOTHESES AND AIMS

The idea this work is based on is that, during classroom interaction, students acquire, through a process of cognitive apprenticeship (Collins *et Al.* 1989), teachers' attitudes and behaviours. This led us to the following two hypotheses: **(a)** The teacher must orient his/her actions to foster students' construction of the three skills (mentioned above) which are fundamental in the development of reasoning by means of algebraic language; **(b)** The teacher has to serve as a *role model* in leading his/her students to a gradual and conscious acquisition of these skills. The research aims related to these hypotheses are: **(1)** To study teachers' attitudes and behaviours in leading their students to the construction of proofs by means of algebraic language and to highlight **(a)** the productivity or negativity of their interventions, pointing out the typical behaviours of an *aware and effective* teacher, and **(b)** the effects of their approach on students, from the point of view of both awareness shown and competencies acquired. **(2)** To identify the specific characteristics of a teacher which effectively act as a role model for his/her students and to propose a first characterisation of the theoretical construct of *teacher as a role model*.

METHODOLOGY

The class-based work was articulated through small-groups activities, collective discussions and individual tests. The data being analysed were students' written productions and the transcripts of the audio-recordings of both small-groups and whole class activities. Each transcript was analysed from different points of view, depending on the typology of activity being analyzed. The analysis of transcripts of collective discussions has been performed by highlighting: (1) weaknesses and strengths of the discussion, with reference to the three key components in the development of proofs in ENT; and (2) the role played by the teacher as a "stimulus" to foster an approach to algebra as a tool for thinking, and, at the same time, as a "model" and "guide" in the construction of reasoning. Transcripts concerning small groups activities were analysed with reference to: (1) weaknesses and strengths of the discussion, with reference to the three key components in the development of proofs in ENT; and (2) the link between types of approach proposed by the teacher in previous activities and types of approach chosen by students, with particular reference to the meta-reflections they propose. Space limitations do not allow us to dwell on the analysis of small-group activities; therefore, we will only report the analysis of a discussion led by an effective teacher.

AN EXAMPLE OF TEACHER ACTING AS AN EFFECTIVE ROLE MODEL

We report here an excerpt from a collective discussion, referring to the construction of the proof of the following statement: *If b is an odd number, the expression 3b represents an odd number*. It is important to stress that, aiming at fostering students' transition from verbal argumentation to algebraic proof, we chose to propose, since the very beginning of the path, an algebraic approach to the justification of statements, even when a verbal one could appear more simple to be adopted. We chose to carry out an in-depth analysis of this particular discussion because it reveals both how the teacher (T) tries to lead pupils to develop those skills that play an essential role in the development of reasoning by means of algebraic language and the positive effects of her approach on students.

During the first part of the discussion, all students agree to formalize the hypothesis through the equality $b=2x+1$ and spontaneously suggest to substitute $2x+1$ instead of b in the expression $3b$. Therefore T writes $3b=3(2x+1)$ on the blackboard. The following excerpt refers to the subsequent part of the discussion.

1 T: What can I do then? (*Silence*) How can you highlight that a number is odd?

2 M: You take a multiple of 2 and add it to 1…

3 T: Does this [*T points* $3(2x+1)$] look like a multiple of 2 added to 1?

4 A: Yes, and then it is multiplied by 3!

5 T: Well... all I have now is 3 times something… you said that my aim is to highlight a multiple of 2 added to 1! Does this look like a multiple of 2 added to 1?

6 A: You can write 3 times $2x$ in brackets, plus 1! [T *writes* $3(2x+1)=3\cdot(2x)+1$]

7 T: Are these two expressions equivalent? [T *points to* $3(2x+1)$ *and* $3\cdot(2x)+1$]

8 Chorus: No!

9 T: Watch out, then! I must carry out transformations which lead me to an expression equivalent to the one I started from!

10 S: You can write 3 times, open bracket, $2x$ in brackets, plus 1, closed bracket [T *writes* $3(2x+1)= 3[(2x)+1]$]

11 M: Why would you do that? It's the same thing!

12 T: Let's keep our aim in mind... What I want to do is to highlight a +1, but in the main expression. Here [T *points to* $3[(2x)+1]$] we have highlighted a +1, but it is inside this factor…

13 A_N: Since we can perform calculations, we could obtain $6x+3$… [T *writes* $3(2x+1)=6x+3$]… Then it becomes $6x+2$, in bracket, then +1 [T *writes* $3(2x+1)=6x+3=(6x+2)+1$]

14 T: She said, "If I perform calculations, I obtain $6x+3$". What is her aim, then? To show that it is possible to highlight a +1. Can we see, now, that it represents an even number added to 1? [T *points to* $(6x+2)+1$]

15 Chorus: Yes!

16 T: Couldn't we do something to highlight this property even more?…

17 M: We can factorize the expression in brackets! Yes! We can take out 2!

18 T: Let's do that… So we have 2 times $3x+1$, plus 1.
[T *writes* $3b=3(2x+1)=6x+3=(6x+2)+1=2(3x+1)+1$]

19 T: Look, now. We've started from $3b$ and we obtained 2 times something plus 1. As M said, an even number plus 1 always gives…

20 S: An odd number.

21 T: So we have accurately proved the statement! Can you follow this reasoning?

22 Chorus: Yes!

Analysis of the excerpt

At the beginning of the discussion T asks how to proceed (line 1), playing the role of an *investigating subject* and acting as a part of the class in the "research" work being activated. In this way, she lets the class guide the activity, although she remains the point of reference for the discussion. Since she gets only silence back, T asks her students whether the constructed expression [$3(2x+1)$] explicitly highlight the property of representing an odd number (line 1). Thus, she becomes an *activator of anticipating thoughts*. Afterward, since her aim is to lead students to syntactically transform the expression they constructed in a way that it could highlight the property of representing an odd number, without relying on any verbal argument, T *stimulates* them through a more explicit question (line 3). Because of A's difficulties (line 4) in following the line of reasoning proposed by T, she decides to *provoke* both a correct interpretation of the examined expression and an effective approach to the manipulations to be carried out, making, once again, the aim of the activity explicit (line 5). Students seem to have followed T's indications, but they have difficulties in identifying what kind of treatments should be applied on the expression in order to better highlight that it represents an odd number. The *anticipating thought* she tried to activate in her students ("I need to highlight the sum between an even number and 1") *prevails on their ability to control the manipulations* they operate, *leading them to perform erroneous treatments* [$3(2x+1)=3\cdot(2x)+1$, line 6]. When A propose an erroneous treatment, T writes on the blackboard the equality he suggests and points out that the two expressions $3(2x+1)$ and $3\cdot(2x)+1$ are not equivalent. Then she puts herself *on a meta level* proposing a comment upon the meaning of the whole activity (the syntactical transformations to be carried out must be allowed, i.e. they must lead to expressions actually equivalent to the starting one, lines 7 and 9). In this way T acts as a *stimulator of reflective attitudes* on the meaning of the activity. She then follow the suggestions proposed by S (line 10), who seems to have lost sight of the aim of the activity: his *anticipating thought* is inhibited *by his need to verify the equivalence between the two expressions* $3(2x+1)$ and $3[(2x)+1]$. Therefore, T's responsibility becomes that of *fostering* in students *an harmonic balance between semantic aspects* (interpreting the expressions in the activated conceptual frames and activating an anticipating thought in relation to the aim of the activity) *and syntactical aspects* (controlling the correctness of the suggested treatments). The meta-comment on the meaning of the activity proposed by M (line 11) highlights that the student has seized the stimuli given by T: his exclamation, "It's the same thing!", does not refer to the equivalence between the two expressions, but to the fact that they both activate the same conceptual frame ('being multiple of'), wherein they have the same interpretation. In order to guide her class toward the right direction, T puts herself once again on a meta-level to draw students' attention to the objective of the syntactical manipulations being performed (line 12), assuming the role of an *activator of anticipating thoughts*. An "indicator" of this typical role played by T is her frequent use of the term "aim". At this point, A_N, showing to have understood T's suggestion, proposes to transform the expression *3*(*2x+1*) into an *additive form* (line 13), analyzes the obtained expression [6x+3] into a new conceptual frame ('even-odd') and suggests to perform the treatment which allows to highlight the addend +1. In this way she displays a good semantic control in carrying out syntactical manipulations under the guidance of a correct anticipating thought. T (line 14) goes back to A_N's line of reasoning (*re-phrasing*, in the words of Anghileri, 2006) to stimulate a moment of reflection on the effectiveness of the approach adopted by the student. This highlights another important role played by the teacher: she acts *as a guide who leads students to the identification of effective models to refer to*. When T asks (line 16) to suggest how to transform the expression (*6x+2*)+1 in a way that it could better highlight the property of representing an odd number, M (line 17) interprets his teacher's question within the frame 'even-odd' and suggests the correct treatment to be performed. T closes the discussion (lines 19 and 21), putting herself once more on a meta-level and re-examining the whole process. Her reflections on both the meaning of the performed syntactical transformations and the effectiveness of the obtained expression, together with the continuing stimuli she has given throughout the discussion, are aimed at *inducing a meta-level attitude* also in students. The analysis of the transcripts of small-groups discussions, not documented herein, enabled us to point out how students have successfully adopted the approach proposed by their teacher as a reference model to face the subsequent activities of the didactical path. Such transcripts, in fact, show an actual maturation of both that awareness and those competencies T has tried to develop in her students.

CONCLUSIONS

The analysis of this transcript, together with the analysis of the transcripts of both collective discussions guided by the observed teachers and their students' discussions while working in small-groups, enabled us to single out the *specific characteristics of an effective action of a teacher in serving as a role model for his/her students*. Moreover, it helped us in sketching the profile of an 'effective teacher' and made us develop the idea of introducing a characterization of the theoretical construct of '*teacher as role model'*. The defining elements of this construct are those characteristics that teachers must fulfil. Summarizing, they must: (a) be able to play the role of an "*investigating subject*", stimulating in his/her students an attitude of research on the problem being studied, and acting as an *integral part* of the class in the research work being activated; (b) be able to play the role of a *practical/strategic guide*, *sharing* (rather than transmitting) knowledge with his/her students, and of a *reflective guide in identifying effective practical/strategic models* during class activities; (c) be aware of their responsibility in maintaining a *harmonized balance between semantic and syntactic aspects* during the collective construction of thought processes by means of algebraic language; (d) try to *stimulate and provoke* the enactment of fundamental skills for the development of thought processes by means of algebraic language (being able to translate, to interpret, to anticipate and to manipulate), playing the role of both an *"activator" of interpretative processes* and an *"activator" of anticipating thoughts*; (e) stimulate and provoke *meta-level attitudes*, acting both as an *"activator" of reflective attitudes* and as an *"activator" of meta-cognitive acts*.

We tested this model in the analysis of other discussions guided by teachers during other activities of the planned path: this enabled us to highlight a clear gap between those teachers who are able to *act as role models* in their classes and those who are not; the latter, in the most extreme cases, produce the opposite effect of stimulating a kind of pseudo-structural approach to the use algebraic language as a thinking tool.

References

Anghileri, J. (2006). Scaffolding practices that enhance mathematics learning. *Journal of Mathematics Teacher Education,* 9, 33–52.

Arcavi, A. (1994). Symbol sense: informal sense-making in formal mathematics. *For the Learning of Mathematics*, 14(3), 24–35.

Arzarello, F. & Bartolini Bussi, M.G. (1998). Italian Trends of Research in Mathematics Education: a National Case Study in the International Perspective. In J. Kilpatrick & A. Sierpinska (Eds.), *Mathematics Education as a Research Domain: A Search for Identity* (pp. 243-262). Kluwer: The Netherlands.

Arzarello, F., Bazzini, L., & Chiappini, G. (2001). A model for analyzing algebraic thinking. In R. Sutherland R. *et Al*. (Eds.), *Perspectives on School Algebra* (pp. 61-81). Kluwer: The Netherlands.

Balacheff, N. (1991). The benefits and limits of social interaction: the case of mathematical proof. In A. Bishop *et Al*. (Eds.), *Mathematical knowledge: its growth through teaching* (pp. 175-192). Kluwer: The Netherlands.

Blanton, M.L., Stylianou, D.A. & David, M.M. (2003). The nature of scaffolding in undergraduate students' transition to mathematical proof. In N. Pateman, *et Al.* (Eds.), *Proc. of PME* 27 (vol. 2, pp. 113-120). Honolulu.

Boero, P. (2001). Transformation and Anticipation as Key Processes in Algebraic Problem Solving. In R. Sutherland *et Al*. (Eds.), *Perspectives on School Algebra* (pp. 99-119). Kluwer: The Netherlands.

Collins, A., Brown, J.S. & Newman, S.E. (1989). Cognitive Apprenticeship: Teaching the Crafts of Reading, Writing and Mathematics! In L.B. Resnick (Ed.), *Knowing, Learning, and Instruction*: *Essays in Honor of Robert Glaser* (pp. 453-494). Hillsdale, NJ: Lawrence Erlbaum Associates.

Cusi, A. (2009a). *Problematiche relative all'insegnamento/apprendimento dell'algebra e ruolo del linguaggio algebrico nell'approccio alla dimostrazione in ambito aritmetico: competenze/consapevolezze dell'allievo ed azione dell'insegnante*. Unpublished PhD dissertation: Università degli studi di Modena e Reggio Emilia.

Cusi, A. (2009b). Interrelation between anticipating thought and interpretative aspects in the use of algebraic language for the construction of proofs in elementary number theory. *Proc. CERME 6*. Lyon. (to appear)

Duval, R. (2006). A cognitive analysis of problems of comprehension in a learning of mathematics. *Educational Studies in Mathematics*, 61, 103–131.

Leikin, R. & Dinur, S. (2007). Teacher flexibility in mathematical discussion. *Journal of Mathematical Behavior*, 26, 328–347.

Leont'ev, A.N. (1978). *Activity, Consciousness and Personality*. Englewood Cliffs: Prentice-Hall.

Martin, T.S., Soucy McCrone, S.M. Wallace Bower, M.L. & Dindyal, J. (2005). The interplay of teacher and student actions in the teaching and learning of geometric proof. *Educational Studies in Mathematics*, 60, 95–124.

Schoenfeld, A.H. (1992). Learning to think mathematically: Problem solving, metacognition, and sense-making in mathematics. In D. Grouws (Ed.), *Handbook for Research on Mathematics Teaching and Learning* (pp. 334-370). New York: MacMillan.

Schwarz, B.B., Hershkowitz, R. & Azmon, S. (2006). The role of the teacher in turning claims to arguments. In J. Novotná *et Al*. (Eds.), *Proc. of PME 30* (vol. 5, pp. 65-72). Prague.

Sfard, A. & Linchevski, L. (1994). The gains and the pitfalls of reification. The case of algebra. *Educational Studies in Mathematics*, 26(2-3), 191-228.

Sherin, M.G. (2002). A balancing act: developing a discourse community in a mathematics classroom. *Journal of Mathematics Teacher Education*, 5, 205–233.

Skemp, R.R. (1976). Relational Understanding and Instrumental Understanding. *Mathematics Teaching*, 77, 20–26.

Vygotsky, L. S. (1978). *Mind in society: The development of higher mental processes*. Cambridge, MA: Harvard University Press.