DEVELOPING EIGHTH GRADERS' CONJECTURING AND CONVINCING POWER ON GENERALISATION OF NUMBER PATTERNS

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This study orchestrates thirty-four eighth graders in a pattern-finding activity of a two week extra curriculum programme to develop conjecturing and convincing power. The activity is designed by applying the binary system to construct six tables with numbers from 1 to 63. A student is asked to select a number between 1~ 63, and check if the number appears in each of the six tables, then the teacher can tell the number accurately. After playing the game for several rounds, the students are divided into six groups to discuss the hidden patterns as well as propose conjectures and make justification. The results clarify that students could not only apply mental representations for generalising but propose impressive justifications to make others convinced during the cognitive process of conjecturing. Besides, their dispositions towards mathematics learning also alter significantly compared with the attitudes they possessed formerly in the regular math classes.

THEORETICAL BACKGROUND

Prominences of features of pattern and generalisation have been outlined by many researchers (Dörfler, 1991; Mason, 1996; Polya, 1954), these features could be a corner stone of numerical, early algebra and algebraic thinking. Mathematics is perceived as the science of pattern and relationship (AAAS, 1990). Consequently, pattern-finding tasks could be a crucial factor for learning of mathematics. Carraher and Schliemann (2007) point out that a pattern is not necessarily a well-defined mathematical object with clear connections to other mathematical objects; rather, it could be considered as a coherent research base in early algebra. Additionally, exploring patterns, relations and functions is an essential focus of algebra (NTCM, 2000).

Researchers acknowledged that pupils' generalising ability could be facilitated efficiently whilst they engage in solving the pattern problems (Becker & Rivera, 2007). Suggestions of some empirical studies also support this viewpoint. Hargreaves, Shorrocks-Taylor and Threlfall (1998) have tested 315 children aged between 7 and 11 years who participated in several number patterning activities, the research results reveal that children could not merely make generalisations with specific strategies, but evolve different cognitive processes.

Actually, generalisations are both objects for individual thinking and means for communication (Dörfler, 1991). Above all, conjecturing is an ongoing process which

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is built on specialising and generalising as an ascent and descent (Polya, 1954). Generalising and specialising are two sides of a coin, in accordance with this view point, Mason (2002) points out two perceptions particularly, which are *seeing the particular in the general* and *seeing the general through the particular*. For that reason, pattern-finding tasks in generalisation can be acknowledged as an important activity for getting students involved in a conjecturing atmosphere. Despite that the significance of conjecturing has been recognised by plentiful researchers (Lakatos, 1976, 1978; Mason, Burton, & Stacey ,1985; Davis, Hersh, & Marchisotto, 1995), it could be recognised that evolving a conjecturing process in patterning approaches is one thing, justifying it for convincing others is quite another. Mason (2002) further states that once a conjecture is made, it needs to be challenged, justified, and possibly reconstructed. As a result, conjecturing accommodates fruitful opportunities for reasoning in behalf of justifying conjectures. Empirical studies have already approved this stance. For instance, Lannin (2005) has held a study with 25 sixth-grade students as they approached patterning tasks in which they were required to develop and justify generalisations. The research result shows that students have demonstrated remarkable abilities to construct generalisations with using different strategies of different justifications to construct and justify the same generalisation.

Lin (2006) stresses that a good lesson must provide opportunities for learners to think and construct actively, in addition, conjecturing is not merely the core of mathematising, but the driving force for mathematical proficiency. According to such standpoint, we organise this study to examine how pupils develop conjecturing and convincing power on reasoning and justifying the generalisation of number patterns.

METHODOLOGY

The study was conducted at a junior high school located in the suburb of central Taiwan. Thirty-four 8^{th} graders participated in a two-week extra curricular programme aimed to foster the power of conjecturing and justifying. According to the students' performances on the previous monthly exam, most of them not only carried out barely below the average, but behaved towards cold end of affect (in the sense of Mandler, 1989).

This pattern-finding activity was purposefully designed to support student development of conjecturing and convincing power. This activity included six tables of numbers (Figure 1) from 1~63 that were constructed by transition of number systems from base-10 to base-2. For instance, decimal number 10 could be represented as the binary number 1010, that is, number 10 is equal to $(1)*2^3+(0)*2^2+(1)*2^1+(0)*2^0$. Among the six tables, table 1 contains the binary numbers with the first digit is 1. The rest may be deduced by analogy, such as table 5 contains the binary numbers with the fifth digit is 1. Hence number 10 can be found in tables 2 and 4 only. The research subjects had not learned the binary system. Hence, randomly placing the numbers in each table but not sequencing them in order might increasing the degree of difficulty of observing the pattern was increased.

The activity started with a game asking a student to select a number from 1~63 and bear it in mind firstly, and then showing the student these six number tables sequentially for she/he to examine whether the selected number is in the table or not. In the end, the teacher notified the student the accurate number she/he had selected. After playing the game for several rounds, students were divided into six heterogeneous groups for working together to find out the hidden patterns, and asked to cooperate together for answering the following questions: (1)What properties can you find from these six tables? (2)Can you induce any generality of the tables from the properties you found? (3)Can you propose any conjecture of how the game works and justify your conjecture? In the end, each group assigned a representative to address their result to the class and answer the students' or teachers questions. After the class, representatives of different groups were interviewed to illustrate their group works based on the worksheets they filled as a record. The whole process of the activity was video-taped, and all these qualitative data were analysed and triangulated.

No. 1

17	19	61	43	25	27	57	31
1	51	37	7	9	11	13	15
33	35	5	39	29	23	45	47
49	3	53	55	41	59	21	63

No. 2

50	51	10	43	58	15	62	63
2	27	6	7	54	22	14	38
18	19	11	23	26	3	30	31
34	35	59	39	42	55	46	47

No. 3

15	5	44	37	12	13	39	4
20	21	61	23	28	29	38	53
52	31	54	47	60	22	62	63
36	7	30	14	6	45	46	55

No. 4

46	13	26	63	28	11	30	31
12	62	42	43	44	45	29	47
56	8	58	59	60	61	41	27
57	9	10	24	40	25	14	15

No. 5

61	17	62	60	20	21	22	23
48	49	50	19	52	53	57	55
24	25	26	27	28	29	58	31
56	54	30	59	51	16	18	63

No. 6

58	33	59	32	36	51	41	39
40	38	42	43	56	48	34	47
63	49	50	37	52	57	54	55
44	53	61	46	60	35	62	45

Figure 1. Number tables for pattern-finding activity

RESULTS AND DISCUSSION

Specialising and generalising, as well as *conjecturing and convincing* are two main subjects of learner's power towards mathematics learning (Mason & Johnston-Wilder). Hence, through the perspectives of these two subjects, students' qualitative data were synthesised and elaborated.

Specialising and Generalising

For the particular cases the students proposed, most of them picked a number from the tables firstly, and then counted the frequency it appeared in these six tables. According to the pattern of the frequencies of different numbers appearing in the tables, students could express the generality from the particular cases. For example, Mike introduced the number pattern which his group found:

Interviewer: How did you find the properties of these tables?

Mike:	We will take any number from the number tables in order to find the frequency which it appears in different tables.
Interviewer:	Why?
Mike:	Because of that, we could outline the pattern of these numbers.
Interviewer:	Can you show me how it works?
Mike:	Take11 as an example. It appears in tables 1, 2 and 4. If someone tells me the number which appears in tables 1, 2 and 4, since only 11 appears in these tables, I can eliminate the others and say the answer is 11.
Interviewer:	Can you tell me the secret?
Mike:	We just rearranged the numbers sequentially in each table, and then observed the frequency of a number is the key to find the pattern.

Apart from Mike's group, George's group employed the similar strategy and took the numbers of even and odd into consideration. They marked all the even numbers in each table (as figure 2), and tried to find the patterns of the odd and even numbers through counting the frequencies. The following transcription is quoted from the interview with George:

Interviewer:	Can you tell me what your group found from the tables?
George:	All the numbers in table 1 are odd numbers and the other tables comprise odd and even numbers half and half.
Interviewer:	What do you mean then?
George:	That is, except table 1, the frequency of odd numbers is in concert with even numbers.

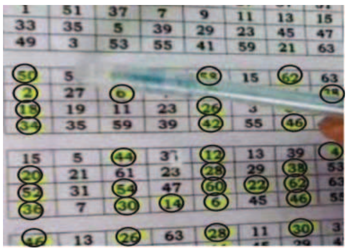

Figure 2. George's group's patterning strategy
(Numbers circled are originally marked by highlighter.).

Examining the students' different generalising strategies, we also noticed that some groups applied the idea of checklist for generalising the number patterns (as figure 3). It makes an echo to the finding of Filloy, Rojano and Rubio (2001) that checklist serves as a tool for helping students move from focusing on a specific example to describing general relationships. Above all, using checklist for students is a facile approach which is also effective in observing the hidden patterns. As in the case of this study, students could go a step further to see the relationship among the numbers.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
1	O	X	O	X	O	X	O	X	O	X	O	X	O	X	O	X	O
2	X	O	O	X	X	O	O	X	X	O	O	X	X	O	X	X	X
3	X	X	X	O	X	O	O	O	X	X	X	O	O	O	O	X	X
4	X	X	X	X	X	X	X	X	O	O	O	X	X	O	O	X	X
5	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	O	O
6	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X

Figure 3. Expressing the generality by using the checklist.

Jessica's group was the one who drew a checklist and then detected a crucial pattern and specific regularity of each number table. Based on these critical discoveries, they were able to propose a conjecture which totally fitted the goal of this activity. The following is the transcription from the interview with Jessica:

Interviewer: Can you tell me what your group found from the tables?

Jessica: We found that 1, 2, 4, 8, 16 and 32 are the first numbers in these six tables after rearranging the numbers sequentially.

Interviewer: Your group drew a checklist of the numbers [on their worksheet], can you explain about it?

Jessica: When drawing a checklist, we found some interesting regularity of the tables. For example, table 1 comprises the odd numbers and starts with number 1; table 2 comprises two successive integers and starts with number 2; table 3 comprises four successive integers and starts with number 4; table 4 comprises eight successive integers and starts with number 8; table 5 comprises sixteen successive integers and starts with number 16; table 6 comprises thirty-two successive integers and starts with number 32.

Interviewer: Any further ideas you got?

Jessica: Yes, we noticed that these six numbers [what she meant are 1, 2, 4, 8, 16, and 32] are powers of 2. Additionally, any two of these numbers cannot appear in the same table. That's why there are six tables in this game.

From the students' responses (especially the above examples), it might suggest that this pattern-finding activity could not only provide opportunities for students to think actively, but also create a proper circumstance for students to evolve their cognitive processes. Moreover, all of the cases clarify that students could generate mental objects for thinking through observing and manipulating the number tables and representing for communication within the cooperation with peers in the pattern-finding activity. It coincides with the viewpoint of Dörfler (1991) that generalisation is not only an object for thinking but a tool for communicating.

Conjecturing and Convincing

To make a conjecture in the pattern-finding activity is one thing, however, to make others persuaded is another. Before persuading others, making oneself persuaded is essential. Therefore, together with observing the patterns through specialising and generalising, the students were also encouraged to propose conjectures based on the patterns they found, and to justify their conjectures to convince their peers.

We take Jessica's group as an example first since they proposed a strong conjecture but only applied some particular examples to make their justification. It might because they have not learned formal mathematical proofs yet.

(continued from the episode quoted formerly on this page)

Interviewer: Based on this discovery, did your group make any conjecture?

Jessica: We would guess that all the numbers from 1 to 63 can be expressed as a sum of 1, 2, 4, 8, 16 or 32.

Interviewer: How can you justify the conjecture?

Jessica: For instance, number 5 is equal to 4+1, number 9 is equal to 8+1, number 21 is the sum of 16+4+1, 61 is the sum of 32+16+8+4+1 and so on.

Interviewer: Can you explain to me how the game works?

Jessica: If the number you choose appears in table 1, then 1 is added. If it appears in table 2, then 2 is added. If it appears in table 3, then 4 is added. By the same rule, if it appears in table 6, then 32 is added.

Interviewer: Can you elaborate it more clearly?

Jessica: You just check a number, say, 11. Since 11 appears in tables 1, 2 and 4, so you have to count 1+2+8 which is 11.

Another example should be Tiffany's group. They noticed that the sum of 1, 3, 8, 15 and 28 is 55, so they made a conjecture that any number between 1 and 63 can be a sum of these five numbers [1, 3, 8, 15 and 28]. However, they could not apply it to most of the numbers. Until hearing Jessica's group's report, they seemingly suddenly saw the light, and rectified their answer.

Interviewer: What conjecture did your group make?

Tiffany: We noticed that the sum of 1, 3, 8, 15 and 28 is 55, so we speculated that any number can be made by these five numbers. But it seems not working with other numbers. Until Jessica addressed her group's result, we found we selected wrong numbers.

Interviewer: So, what did you do then?

Tiffany: We checked that 55 can be the sum of numbers 1, 4, 8, 16 and 32 [it should be 1, 2, 4, 16 and 32] as well. We also found that the sum of 1, 2, 4, 8, 16 and 32 is 63, which is the biggest number in every table.

Although all the six groups of students could find some generality of these number tables through observing and manipulating, most of them could not propose appropriate conjectures based the generality they found. However, from the case of Jessica's group, a suitable activity is still able to help the students develop the ability of conjecturing and convincing. Besides, through the cooperation with peers in group and sharing findings with the whole class, the students have more opportunities to construct their own understanding and thinking actively. Moreover, in contrast with the majority of students' former disposition, interestingly, they exhibited rather positive attitudes mathematics learning in this problem-solving activity.

CONCLUSION

The results of this study disclose that, through observing and manipulating, most of the students could develop the ability of specialising and generalising. Based on the results of specialising and generalising, some students might continue to make proper conjectures and justify them on their own. In the whole learning process, the collaboration within peers plays a vital role. Besides, the teacher's guidance is also crucial when students are stuck with a choke point, or miss some important clues. Furthermore, this activity seems to improve the research subjects' attitudes towards learning mathematics as they possessed very low learning motivation. Therefore, we could suggest this kind of activities should be good for the preparation of introducing a new mathematical concept. For instance, the activity designed in this study could be a good pioneering activity for introducing the concept of binary system.

This study offers some empirical evidence of Lin's claim in the plenary speech presented in APEC-TSUKUBA International Conference held in Tokyo, that "*A good lesson must provide opportunities for learners to think and construct actively*", and "*Conjecturing is the centre and pivot of all phases of mathematics learning – including conceptualising, procedural operating, problem solving and proving, and provides the driving force for developing these phases of mathematics learning*" (Lin, 2006). Above all, learning with understanding is essential for enabling students to solve the new kinds of problems they will inevitably face in the future (NCTM, 2000). According to their former learning experiences, most of the students are excessively focused on instrumental understanding without any extra energetic thinking. Consequently, the study is orchestrated for developing pupils' conjecturing and convincing power as well as for facilitating students' learning of mathematics towards relational understanding (in the sense of Skemp, 1987).

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