# TOWARDS A COMPREHENSIVE FRAME FOR THE USE OF ALGEBRAIC LANGUAGE IN MATHEMATICAL MODELLING AND PROVING

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*In this paper we consider the use of algebraic language in modelling and proving. We will show how a specific adaptation of Habermas' construct of rational behaviour allows to describe and interpret several kinds of students' difficulties and mistakes in a comprehensive way, provides the teacher with useful indications for the teaching of algebraic language and suggests further research developments.*

## INTRODUCTION

According to Habermas' definition (see Habermas, 2003, Ch. 2), a rational behaviour in a discursive practice can be characterized according to three inter-related criteria of rationality: *epistemic* rationality (inherent in the conscious control of the validity of statements and inferences that link statements together within a shared system of knowledge, or theory); *teleological* rationality (inherent in the conscious choice and use of tools and strategies to achieve the goal of the activity); *communicative* rationality (inherent in the conscious choice and use of communication means within a given community, in order to achieve the aim of communication).

In our previous research we have dealt with an adaptation of Habermas' construct of rational behaviour in the case of conjecturing and proving (see Boero, 2006; Morselli, 2007; Morselli & Boero, 2009 - to appear). In this paper we focus our interest on the use of algebraic language in proving and modelling. Algebraic language will be intended in its ordinary meaning of that system of signs and transformation rules, which is taught in school as a tool to generalize arithmetic properties, to develop analytic geometry and to model non-mathematical situations (in physics, economics, etc.). In particular, for what concerns modelling (see Norman, 1993, and Dapueto & Parenti, 1999) algebraic language can play two kinds of roles: a tool for proving through modelling within mathematics (e.g. when proving theorems of elementary number theory) - *internal modelling*; or a tool for dealing with extra-mathematical situations (in particular to express relations between variables in physics or economy, and/or to solve applied mathematical problems) - *external modelling*.

Our interest for considering the use of algebraic language in the perspective of Habermas' construct depends on the fact that our previous research (Boero, 2006; Morselli, 2007) suggests that some of the students' main difficulties in conjecturing and proving depend on specific aspects (already pointed out in literature) of the use of algebraic language, which make it a complex and demanding matter for students.

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In particular, we refer to: the need of checking the validity of algebraic formalizations and transformations; the correct and purposeful interpretation of algebraic expressions in a given context of use; the goal-oriented character of the choice of formalisms and of the direction of transformations; the restrictions that come from the need of following taught communication rules, which may contradict private rules of use or interfere with them. In this paper, we will try to show how framing the use of algebraic language in the perspective of Habermas' theory of rationality: first, provides the researcher with an efficient tool to describe and interpret in a comprehensive way some of the main difficulties met by students when using algebraic language; second, provides the teacher with some useful indications for the teaching of algebraic language; third, suggests new research developments.

## ADAPTATION OF HABERMAS' CONSTRUCT OF RATIONAL BEHAVIOUR TO THE CASE OF THE USE OF ALGEBRAIC LANGUAGE

### Epistemic rationality

It consists in:

- *modelling requirements*, concerning coherency between the algebraic model and the modelled situation: control of the correctness of algebraic formalizations (be they *internal* to mathematics - like in the case of the algebraic treatment of arithmetic or geometrical problems; or *external* - like in the case of the algebraic modelling of physical situations) and interpretation of algebraic expressions;

- *systemic requirements* in the use of algebraic language and methods. In particular, these requirements concern the manipulation rules (syntactic rules of transformation) of the system of signs usually called algebraic language, as well as the correct application of methods to solve equations and inequalities.

### Teleological rationality

It consists in the conscious choice and finalization of algebraic formalizations, transformations and interpretations, according to the aims of the activity. It includes also the management of the writer-interpreter dynamics (Boero, 2001): the author may write an algebraic expression under an intention and, after, interpret it in a different goal-oriented way, by "seeing" new possibilities in the written expression.

### Communicative rationality

In the case of algebraic language we need to consider not only the communication with others (explanation of the solving processes, justification of the performed choices, etc.) but also the communication with oneself (in order to activate the writer-interpreter dynamics). Communicative rationality requires the user to follow not only community norms concerning standard notations, but also criteria for easy reading and manipulation of algebraic expressions.

**Some comments**

We are aware of the existence of several analytical tools to deal with the teaching and learning of algebraic language. In our opinion, Arcavi's work on Symbol sense (Arcavi, 1994) offers the most comprehensive perspective for the use of algebraic language. With different wordings, it includes concerns for teleological rationality and some aspects of epistemic rationality. Comparing our approach with Arcavi's elaboration, we may say that we add the communicative dimension of rationality. We will see how it will allow us to account for: the possible tension between private rules of communication in the intra-personal dialogue, and standard rules; and the interplay between verbal language and algebraic language. Moreover we will see how our distinctions between the epistemic dimension and the teleological dimension, and between the modelling requirements and the systemic requirements of epistemic rationality allow to deal with the tensions and the difficulties that can derive from their coordination.

In order to justify a new analytic tool in Mathematics Education it is necessary to show how it can be useful in describing and interpreting students' behaviour, and/or in orienting and supporting teachers' educational choices, and/or in suggesting new research developments. We will try to show it in the following Sections.

## DESCRIPTION AND INTERPRETATION OF STUDENTS' BEHAVIORS

The following examples are derived from a wide corpus of students' individual written productions and transcripts of *a posteriori* interviews, collected for other research purposes in the last fifteen years by the Genoa research team in Mathematics Education. In particular, we will consider three categories of students: (a) 9$^{th}$ grade students who are approaching the use of algebraic language in proving; (b) students who are attending university courses to become primary school teachers; (c) students who are attending the third year of the university course in Mathematics.

A common feature for all the considered cases is that the individual tasks require not only the solution, but also the explanation of the strategies followed to solve the problem. Each individual task was followed by *a posteriori* interviews.

**EXAMPLE 1: 9$^{th}$ grade class**

The class (22 students) was following the traditional teaching of algebraic language in Italy: transformation of progressively more complex algebraic expressions aimed at "simplification". In order to prepare students to the task proposed by the researcher, two examples of "proof with letters" had been presented by the teacher; one of them included the algebraic representation of even and odd numbers.

THE TASK: "Prove with letters that the sum of two consecutive odd numbers is divisible by 4".

Here we report some recurrent solutions (in parentheses the number of students who performed such a solution; note that "**d**ispari" means "odd" in Italian)

E1 (4 students): $d+d=2d$

In this case, we can observe how the *systemic requirements of epistemic rationality* are satisfied (algebraic transformation works well), while the *modelling requirements* fail to be satisfied (the same letter is used for two different numbers).

E2 (8 students): $d+d+2=2d+2$

In this case, both the *systemic and the modelling requirements of epistemic rationality* are satisfied, but the requirements inherent in *teleological rationality* are not satisfied: students do not realize that the chosen representation does not allow to move towards the goal to achieve (because the letter *d* does not represent in a transparent way the fact that *d* is an odd number) and do not change it.

E3 (5 students): $d=2n+1+dc=2n+1+2n+1+2=4n+4$ (or similar sequences)

We can infer from the context (and also from some a-posteriori comments by the students) that "dc" means "**d**ispari **c**onsecutivi" (consecutive odd numbers).

In this case *epistemic rationality* fails in the first and in the second equality, but *teleological rationality* works well: the flow of thought is intentionally aimed at the solution of the problem; algebraic transformations are used as a calculation device to produce the conclusion (divisibility by 4).

**EXAMPLE 2: University entrance, primary school teachers' preparation** The following task had been preceded by the same task of the Example 1, performed under the guide of the teacher. 58 students performed the activity.

THE TASK: Prove in general that the product of two consecutive even numbers is divisible by 8

Very frequently (about 55% of cases) students performed (without comments) a long chain of transformations, with no outcome, like in the following example:

E4: $2n(2n+2)=4n^2+4n=4(n^2+n)=4n(n+1)=4n^2+4n=n(4n+4)$

In this case, we see how both requirements of *epistemic rationality* are satisfied: *modelling requirements* (concerning the algebraic modelling of odd numbers and even numbers); and *systemic requirements* (correct algebraic transformations). The difficulty is inherent in the lack of an interpretation of formulas leaded by the goal to achieve, thus in *teleological rationality*. The student gets lost, even if the interpretation of the fourth expression would have provided the divisibility of $n(n+1)$ by 2 because one of the two consecutive numbers n and n+1 must be even.

In the following case, both *modelling* and *systemic requirements* are not satisfied: the same letter is used for two consecutive even numbers (note that "**p**ari"means "even" in Italian) and the algebraic transformation is affected by a mistake.

E5: $p*p=2p^2$, divisible by 8 because p is divisible by 2 and thus $p^2$ is divisible by 4.

The student seems to work under the pressure of the aim to achieve: having foreseen that the multiplication by 2 may be a tool to solve the problem, she tries to justify it by considering the juxtaposition of two copies of p that generates "2". Indeed in the interview the student said that she had made the reasoning "p is divisible by 2 and thus $p^2$ is divisible by 4" before completing the expression. In this case we can see how *teleological rationality* prevailed on *epistemic rationality* and hindered it.

We have also found cases like the following one:

> E6: p*(p+2)=p2+2p=8k because p2+2p=8 if p=2

Also in this case, from the *a posteriori* interview we infer that probably the lacks in *epistemic rationality* depend on the dominance of *teleological rationality* without sufficient epistemic control:

> I have seen that in the case p=2 things worked well, so I have thought that putting a multiple 8k of 8 in the general formula would have arranged the situation.

**EXAMPLE 3: The bomb problem - Third year mathematics students**

TASK: A helicopter is standing upon a target. A bomb is left to fall. Twenty seconds after, the sound of the explosion reaches the helicopter. What is the relative height of the helicopter over the ground?

The problem was proposed to groups of third year mathematics students in seven consecutive years. Some reminds were provided about the fact that the falling of the bomb happens according to the laws of the uniformly accelerated motion, while the sound moves at the constant speed of 340 m/s. However no formula was suggested.

The problem is a typical applied mathematical problem, whose solution needs an *external modelling* process. In terms of *teleological rationality*, the goal to achieve should result in the choice of an appropriate algebraic model of the situation, in solving the second degree equation derived from the algebraic model, and in choosing the good solution (the positive one).

The difficulties that students meet consist: in the time coordination of the two movements (it is necessary to enter somewhere in the model the information that the whole time for the bomb to reach the ground and for the sound of the explosion to reach the helicopter is 20 seconds); and in their space coordination (the space covered by the falling bomb is the same covered by the sound when it moves from the ground to the helicopter). Let us consider some students' behaviours.

Most students are able to write the two formulas:

> E7: $s=0{,}5\ gt^2$, $s=340\ t$

They are standard formulas learnt in Italian high school in grades $10^{th}$ or $11^{th}$, in physics courses. About 20% of the students stick to those formulas without moving further. From their comments we infer that in some cases the use of the same letters for space and time in the two algebraic expressions generates a conflict that they are not able to overcome. We can see how general expressions that are correct for each of the two movements (if considered separately) result in a bad model for the whole phenomenon. *Teleological rationality* should have driven formalization under the control of *epistemic rationality*; such control should have put into evidence the lack of the *modelling requirements* of *epistemic rationality*, thus suggesting a change in the formalization. In the reality for those students such an interplay between *epistemic rationality* and *teleological rationality* did not work.

In other cases (about 10% of the sample) the coordination of the two times was lacking, and the idea of coordinating the spaces (together with the formalization of both movements with the same letters) brought to the equation:

E8: $0{,}5\ gt^2 = 340\ t$

with two solutions t=0, t=68 that some students were unable to interpret and use (because 68 is out of the range given by the text of the problem). But other students found the height of the helicopter by multiplying 340x68, with no critical reaction or re-thinking, probably because it is normal that school problems are unrealistic!

Less than 60% of students wrote a good model for the whole phenomenon:

$t_b+t_s= 20$ $\qquad\qquad$ $h=0{,}5gt_b^2=340t_s$

and moved to a second degree equation by substituting $t_s=20\text{-}t_b$ or $t_b=20\text{-}t_s$ in the equation: $0{,}5gt_b^2=340t_s$

Many mistakes occurred during the solution of the equation (mainly due to the management of big numbers). Once two solutions were got (one positive and the other negative), in most cases the choice of the positive solution was declared but not motivated. *A posteriori* comments reveal that the fact that a negative solution is unacceptable (given that the other solution is positive!) was assumed as an evidence, without any physical motivation.

In terms of *epistemic rationality*, three kinds of difficulties arose; they were inherent: first, in the control that the chosen algebraic model was a good model for the physical situation; second, in the control of the solving process of an equation with unusual complexity of calculations (big numbers); third (once the valid equation - a second degree equation - was written and solved), in the motivation of the chosen solution.

In terms of *communicative rationality*, we can observe how (in spite of the request of explaining the steps of reasoning) very few students of both samples were able to justify the crucial steps of the solving process. *A posteriori* interviews revealed that most students who had been unable to justify their choices were sure about their method only afterwards, when checking the positive solution and finding that it was “realistic”, thus putting into evidence a lack in *teleological rationality* (lack of consciousness about the performed modelling choices). However a number of solutions was quite realistic, even if got through a bad system. Many authors of the correct solutions were not able to explain why the other solutions were mistaken. This suggests that lacks in *communicative rationality* (as concerns verbal justification of the validity of the performed modelisation) can reveal lacks in *teleological rationality* (motivation of choices with reference to the aim to achieve) and even in *epistemic rationality* (control of the validity of the steps of reasoning).

## DISCUSSION

As remarked in the second section, the usefulness of a new analytical tool in mathematics education must be proved through the *actual* and the *potential research advances* and the *educational implications* that it allows to get.

### Research advances

In the frame of our adaptation of Habermas' construct, the distinction between *epistemic rationality* and *teleological rationality* allows to describe, analyse and interpret some difficulties (already pointed out in Arcavi's work), which depend on the students' prevailing concern for rote algebraic transformations performed according to *systemic requirements* of *epistemic rationality* against the needs inherent in *teleological rationality* (see E4). Moreover, the distinction in our model between *modelling requirements* and *systemic requirements* of *epistemic rationality* offers the opportunity of studying the interplay between the *modelling requirements* and the requirements of *teleological rationality* (see E7); we have also seen that formalization and/or interpretations may be correct but not goal-oriented (like in E2 and E4), or incorrect but goal-oriented (like in E5, E6 and E8). Together with the other dimensions of rationality, *communicative rationality* allows to describe and interpret possible conflicts between the private and the standard rules of use of algebraic language, and the ways student try to integrate them in a goal-oriented activity (see E3).

Further research work should be addressed to establish what mechanisms (meta-cognitive and meta-mathematical reflections based on the use of verbal language? See Morselli, 2007) can ensure the control of *epistemic rationality* and the intentional, full development of *teleological rationality* in a well integrated way. With reference to this problem, taking into account *communicative rationality* (in its intra-personal dimension, possibly revealed through suitable explanation tasks and/or interviews) suggests a research development concerning the role of verbal language (in its mathematical register: see Boero, Douek & Ferrari, 2008, p.265) in the complex relationships between *epistemic, teleological* and *communicative rationality*.

### Educational implications

We think that the analyses performed in the previous section can provide teachers as well as teachers' educators with a set of indications on how to perform educational choices and classroom actions to teach algebraic language as an important tool for modelling and proving. Some of those indications are not new in mathematics education; we think that the novelty brought by the Habernas' perspective consists in the coherent and systematic character of the whole set of indications.

First of all, the performed analyses suggest to balance (at the students' eyes, according to the didactical contract in the classroom) the relative importance (in relationship with the goal to achieve) of: production and interpretation of algebraic expressions, vs algebraic transformations; and flexible, goal-oriented direction of algebraic transformations, vs rote algebraic transformations aimed at "simplification" of algebraic expressions. These indications are in contrast with the present situation in Italy and in many other countries: teachers' classroom work is mainly focused on algebraic transformations aimed at "simplification" of algebraic expressions. The fact that algebraic expressions are given as objects to "simplify" (and not as objects to build, to transform according to the aim to achieve, and to interpret during and after the transformation process in order to understand if the chosen path is effective and correct or not) has bad consequences on students' *epistemic rationality* and *teleological rationality*. As we have seen, many mistakes occur in the phase of formalization (against the *modelling requirements*), and even when the produced expressions are correct, frequently students are not able to use intentionally them to achieve the goal of the activity (against the *teleological rationality requirements)*.

A promising indication coming from our analyses concerns the need of a constant meta-mathematical reflection (performed through the use of verbal language) on the nature of the actions to perform and on the solving process during its evolution. At present, the only reflective activity in school concerns checking the correct application of the rules of syntactic transformation of algebraic expressions (thus only one component of rational behaviour - namely, the *systemic requirements* of *epistemic rationality* - is partly engaged).

**References**

Arcavi, A. (1994). Symbol sense: Informal sense-making in formal mathematics. *For the Learning of Mathematics*, *14*, 3, 24-35.

Boero, P. (2001). Transformation and anticipation as key processes in algebraic Problem solving. In R. Sutherland et al. (Eds), *Perspectives on School Algebr*a (pp. 99-119). Dordrecht (NL): Kluwer.

Boero, P. (2006). Habermas' theory of rationality as a comprehensive frame for conjecturing and proving in school. *Proc. of PME-30* (vol.2, pp. 185-192).

Boero, P., Douek, N. & Ferrari, P. L. (2008). Developing mastery of natural language. In L. English (Ed.), *International Handbook of Research in Mathematics Education* (pp. 262-295). New York: Routledge.

Dapueto, C. & Parenti, L. (1999). Contributions and obstacles of contexts in the development of mathematical knowledge. *Educational Studies in Mathematics*, *39*, 1-21.

Habermas, J. (2003). *Truth and justification*. Cambridge (MA): MIT Press.

Morselli, F. (2007). *Sui fattori culturali nei processi di congettura e dimostrazione*. PhD Thesis. Università degli Studi di Torino.

Morselli, F. & Boero, P. (2009). Habermas' construct of rational behaviour as a comprehensive frame for research on the teaching and learning of proof. *Proceedings of the ICMI Study 19*, Taipei (to appear).

Norman, D. A. (1993). *Things that Make us Smart*. London: Addison-Wesley.