CONJECTURING AND PROVING IN ALNUSET

Bettina Pedemonte

DiDiMa srl., Istituto per le Tecnologie Didattiche – CNR Genova

This report proposes an approach to algebraic proof. It is based on the use of the AlNuSet system, a dynamic, interactive system to enhance the teaching and learning of algebra, numerical sets and functions in secondary school. This work wishes to show in which way AlNuSet can be used in the educational practice to enhance the teaching and learning of algebraic proof. The research hypothesis is that educational activities performed in AlNuSet favour the combination of the aspects of visibility and invisibility in the approach to proof, making it "transparent".

Key words: Algebra, AlNuSet, Transparency, Conjecturing and proving

PROOF IN ELEMENTARY ALGEBRA

Bibliography on the theme of algebraic proof at secondary school level is sporadic (Healy & Hoyles, 2000). A possible reason is that in the secondary school curricula of many countries, the approach to proof is still taught in the context of traditional geometry (Hanna & Jahnke, 1993). Nevertheless, rigorous proof is generally considered as a sequence of formulae within a given system, each formula being either an axiom or derivable from an earlier formula by a rule of the system. This kind of proof clearly reveals the influence of algebra (Hanna & Jahnke, 1993).

Furthermore, some recent studies (Pedemonte, 2008) show that Algebra seems to be a good domain to introduce proof. In fact, unlike the geometrical case, in Algebra some difficulties students have in the construction of proof seem not to be present. This has been observed when students solve open problems requiring the construction of a conjecture and the production of a proof. Some Italian researchers (Boero, Garuti Mariotti 1996, Garuti e al. 1996, 1998, Mariotti 2001) showed that open problems are suitable for proof learning because *cognitive unity* between argumentation supporting the conjecture and the construction of the proof can be realised. According to *cognitive unity* hypothesis, the argumentation used to construct a conjecture can be used by students in the construction of proof by organising some of the previously produced arguments in a logical chain. This continuity supports the construction of a proof. However, another kind of continuity, the *structural continuity,* exists between argumentation supporting a conjecture and proof (Pedemonte, 2007). This kind of continuity occurs when argumentation and proof have the same structure (abductive, inductive, deductive). Pedemonte (2007) observed that this continuity can be an obstacle for the construction of a geometrical proof: some students do not construct a proof because they are unable to transform abductive steps of argumentation into deductive ones in the proof. When constructing algebraic proof this obstacle seems not to be present (Pedemonte, 2008). Since algebraic proof is characterised by a strong deductive structure, abductive steps in the argumentation activity can be useful in linking the meaning of the letters used in the algebraic proof with numbers used in the argumentation. Thus, the approach to proof in Algebraic domain could be more effective than in Geometry.

In school practice, however, algebra is not usually considered as a way of seeing and expressing relationships but as a body of rules and procedures for manipulating symbols. Normally, students can develop manipulations but are not aware of axioms and theorems they are using in performing it. Thus, algebra is taught and learned as a language and emphasis is given to its syntactical aspects. The actual "vision" of Algebra should be modified to introduce it as a domain for proof - algebra should be considered as a theoretical system where techniques used to make manipulation derive from mathematical axioms and rules.

This paper proposes an approach to algebraic proof. It is based on the use of the AlNuSet system which can be used to propose specific tasks requiring the construction of a conjecture and the production of an algebraic proof. AlNuSet was developed in the context of ReMath (IST - 4 - 26751) EC project for students of lower and upper secondary school (years 12-13 to 16-17). It is constituted by three integrated environments: the *Algebraic Line*, the *Algebraic Manipulator*, and the *Functions.* In this paper we consider two of them: the Algebraic Line (AL) and the Algebraic Manipulator (AM). The AL is an explorative environment to construct conjectures through a motor perceptive approach; the AM is a symbolic calculation environment to produce algebraic proof. The aim of this report is to show how this system can be used to support the teaching and learning of algebraic proof, making proof "transparent".

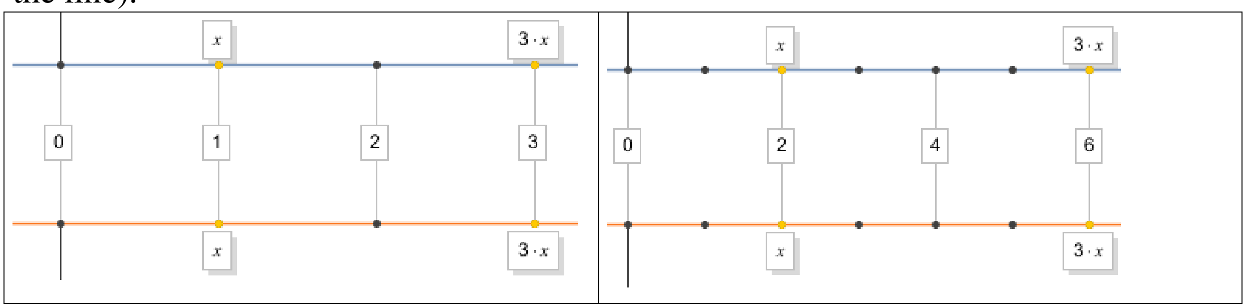
PROOF HAS TO BE "TRANSPARENT"

The role of the proof in the educational practice is not well defined and very often difficulties emerge because some aspects of proof are not explicit for students and they are not well explained by teachers (Hemmi, 2008). Through the notion of "transparency", in her report Hemmi contributes to solve the dilemma to make more or less invisible for students some important aspects concerning proof. The concept of transparency (Lave and Wenger, 1991) combines two characteristics: visibility and invisibility. Visibility concerns the ways that focus on the significance of proof (construction of the proof, logical structure of proof, its function, etc.). Invisibility is the form of "unproblematic interpretation and integration to the activity" (Hemmi, 2008, p. 414). It concerns the proof as a justification of the solution of a problem without thinking it as a proof. It has been underlined that "*Proof as an artifact needs to be both seen (to be visible) and used and seen through (to be invisible) in order to provide access to mathematical learning*" (Hemmi, p. 425). The lack of visibility in the teaching of proof regards the lack of knowledge about proofs techniques, key ideas and proof strategies.

The hypothesis of this study is that AlNuSet can be used in teaching and learning algebraic proofs to make proof more "transparent". The AL can be used to make "visible" some important mathematical concepts that are usually implicit in the algebraic manipulation (the variable, the dependence of an expression from the variable, the meaning of equation, etc.). The AM of AlNuSet can be used in teaching and learning algebraic proofs to make rules and axioms used “visible” in proof processes and to let theoretical aspects usually implicit in algebraic manipulation emerge. The AL and the AM are briefly presented in the following. A more detailed presentation can be found in other reports (Chiappini, Pedemonte, Robotti, 2008; Pedemonte, Chiappini, 2008).

THE ALGEBRAIC LINE OF ALNUSET

The AL of AlNuSet is constituted by two lines[1] where it is possible to insert letters and mathematical expressions involving numbers and letters. These expressions can be inserted (or constructed) and represented as points on the line depending on the mobile point of the variable contained in such expressions. Once an expression has been inserted, dragging the x mobile point, the expression(s) that depend on it move accordingly (i.e. in the figure below the expression 3x moves when x is dragged on the line).



This dynamic characteristic is very important to allow students experience important algebraic concepts - the dependence of the expression from a variable, the meaning of denotation for an expression, the equivalence among expressions, etc. These aspects are detailed in the following.

THE ALGEBRAIC MANIPULATOR OF ALNUSET

The AM of AlNuSet is a structured symbolic calculation environment for the manipulation of algebraic expressions and for the solution of equations and inequalities. Its operative features are based on pattern matching techniques. In the Algebraic Manipulator pattern matching is based on a structured set of basic rules that correspond to the basic properties of operations, to the equality and inequality properties between algebraic expressions, to basic operations among propositions and sets. These rules are explicit for students. They appear as commands on the interface and made active only if they can be applied to the part of expression previously selected. An expression is transformed into another through this set of commands that corresponds to axioms and rules. Students can see the transformation of an expression as the result of the application of a rule on it.

[1] The two lines are used to construct expressions through a geometrical model. It is not possible here to explain in which way they work. In this report we use these lines as a unique line.

TEACHING EXPERIMENT

In this section student's resolution processes of some tasks involving the production of a conjecture and the construction of proof in AlNuSet are analysed. They are taken from a set of data collected from an experiment carried out in a class of 22 students of the Second year of Lower Secondary School (12-13 years old). Empirical data were both qualitative and quantitative and were collected according to different methodologies: written tests, observations and recording of students dialogs. Data were transcribed and translated from Italian into English.

The main aim of this experiment was to analyse the role of AlNuSet in a teaching experiment centred on algebraic expressions and propositions. The experiment lasted 6 weeks, with sessions of two hours per week. The first session of the teaching experiment focused on algebraic expressions. In this report we present results of this session. Students worked in pairs with AlNuSet under the supervision of the teacher and the researcher. Students had not used AlNuSet previously. The teacher presents the software showing some specific technical features. Then she distributes a paper containing the tasks.

Tasks

a) Let x be an integer number. Write an expression for the triple of x.
Represent this expression on the AL. Is your answer correct? Why?
Write an expression for the consecutive of the triple of x. Represent it on the AL and verify your answer.
Consider the expression x+2x+1. Compare this expression with the previous one.
Check your answer using the AL and AM of AlNuSet.

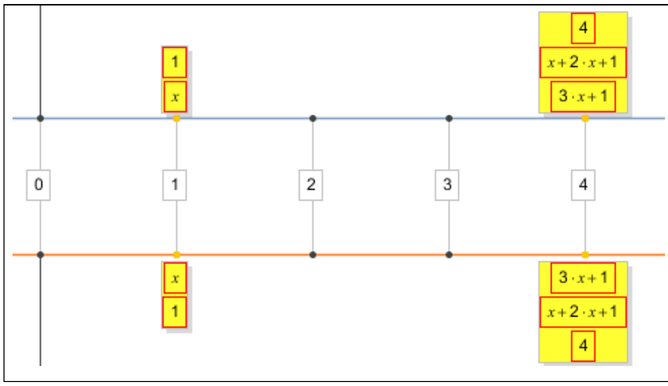
b) Let x be an integer number. Write an expression that represents the quadruple of x increased by 3.
Is there any value of x such that this expression is 27?

c) The teacher asks a student to carry out the following computation:
Think a number, double it, add 6, divide the result by two, and subtract from it the number that you thought initially.
The teacher says:
The result is 3
The teacher proposes the exercise to two other students changing the number to add.
At the end she proposes the following exercise:
If **x** is the initial number, and **a** is the number to add, write an expression to translate the described computation.
Represent the expression on the AL.
What can you observe when you move **x**? What can you observe when you move **a**?
What is the result of the expression? Represent the result on the AL and check your hypothesis.

<u>Tasks a)</u> requires to construct the expression *3x* in the AL and verify that this expression represents the triple of *x*. Moving *x* on the line the expression *3x* moves accordingly. Through a perceptive approach students can see that the point associated to the expression *3x* assumes values that are multiples of 3. In this way, what the expression *3x* denotes is made more explicit. Furthermore, only moving the variable

x it is possible to move *3x* allowing students to experience the expression's dependence from the variable x.

The second part of the task requires to construct the consecutive of the triple of *x* and to compare it with the expression *x+2x+1*. The aim of this second question is to point out the equivalence between the two expressions from a perceptive point of view and not from a formal one. In the AL the equivalence among expressions is represented by a post-it (see figure below). The two expressions *3x+1* and *x+2x+1* belong to a same post-it for each value the variable x assumes on the line.



$x+2\cdot x+1$

$x\cdot 1+2\cdot x+1$

$1\cdot x+2\cdot x+1$

$(1+2)\cdot x+1$

$(3)\cdot x+1$

$3\cdot x+1$

Students can "experience" the equivalence of the two expressions and then they can prove the equivalence in the AM. In the AL, students make visible the equivalence between the two expressions. The focus here is not to prove the equivalence but to experience it. In the AM proof is made explicit - students are obliged to explicit the rules necessary to transform the first expression into the other. This is not obvious because this transformation in a paper and pen environment is usually not treated as a proof; here proof is in general invisible to students.

<u>Analysis of tasks a) results</u>

All students are able to answer the first question: they write the expression *3x*. However it is interesting to observe that students are not able to justify why *3x* is the triple of x.

Some students construct a table, other students tell the 3 times table. Only when they can move the expression on the AL they are able to explicit that the expression *3x* denotes the triple of x because "*the expression assumes only values that are multiples of 3... it probably takes all multiple of 3*" (Sara).

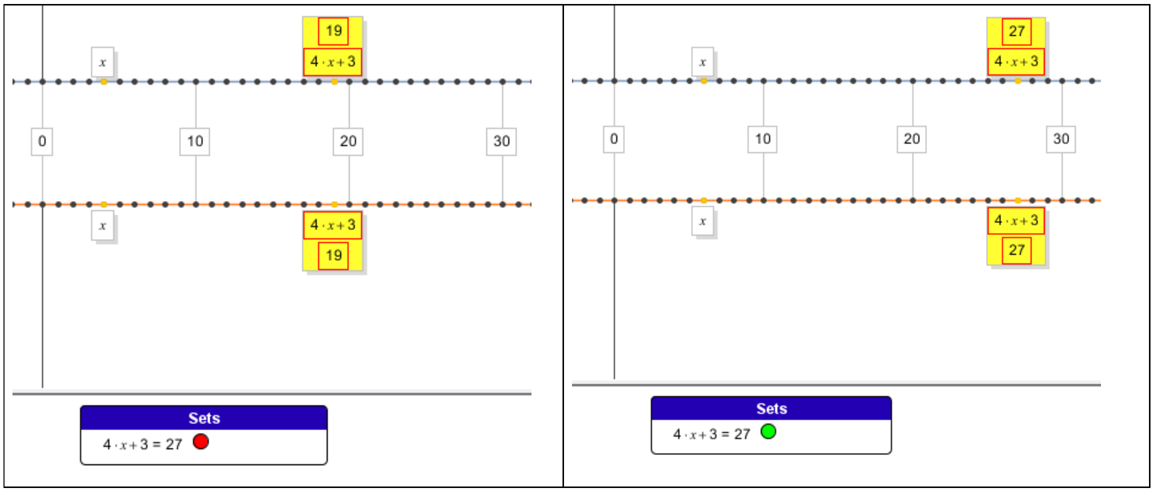
Another interesting aspect that emerged during the exploration was the dependence of the expression *3x* from the variable *x*. When students have to move the expression *3x*, they fail because they move directly the expression and not the variable *x*. Here an example of two students who are not able to move *3x*.

> Francesca: 3x does not move
> Danilo: We are not able to move 3x
> Teacher: Why isn't it moving?
> Francesca: we are trying but with no results…
> Teacher: Why are you moving directly 3x?

Francesca: Because I want to move the expression 3x
Teacher: You should not move directly 3x because this expression is dependent on the variable x
Silence
Teacher: In which way can you move 3x?
Silence
Teacher: you have to move x
Danilo: x??
Danilo moves x
Danilo: Ahh... 3x is dependent on x
Francesca: Ahhh… it was really too difficult for us…

All students write the expression *3x+1* and they insert it on the line. Once more some students try to move the expression directly on the line. The intervention of the teacher is necessary to overcome this obstacle. All students are aware that the two expressions *x+2x+1* and *3x+1* are equal because a calculus rule (*x+2x* is equal to *3x*). Proof is invisible here. On the contrary, in AM, the transformation of an expression into the other becomes visible: many students feel frustrated when they have to prove their calculations with AM. The teacher has to guide them specifying properties they are using (distributive property, insertion of the neutral element, etc.). In AM proof is not spontaneous for students. As highlighted by Hemmy, visibility and invisibility of proof interact in the process of learning and both are needed. It was unexpected for the teacher that after the teaching experiment and during the usual lessons students could explicit these properties also in a paper and pen environment or on the blackboard and when not required. For example, they used to say "*we are applying distributive property*" to replace the usual statement "*we are performing a calculation*". Moreover, the work with AL was important for students to make visible the relation between the value of a variable and the value of an algebraic expression or proposition. The following statement was written by a student "*An expression is dependent from a variable, from the letter that is contained in it. If in the expression there are only numbers, then the expression is not dependent from the variable*" (Carlo).

<u>Tasks b)</u> is an implicit requirement to solve an equation. In AlNuSet (as shown in the next figure) it is possible to solve the equation dragging *x* to move the expression *4x+3* in the point *27*. When *x* is situated on the point *6*, the expression *4x+3* is on the point *27* and the little ball associated to the equation is green. On the contrary, when x is moved on the other values, the little ball is red to show that other points are not a solution for the equation. As a consequence, AL makes available functionalities to solve equations in a non formal way.



This feature helps understand the meaning of equation. Students are usually able to solve the equation through the manipulation rules but they cannot say that the solution makes it true if replaced in the equation.

<u>Analysis of tasks b) results</u>

All students are able to solve this task, and a pair of them recognises that they are solving an equation. However, it is interesting to observe that many solve the equation on the paper before the exploration on the line. Proof is invisible in the paper. In the AL some important mathematical concepts useful to understand the proof may become visible. Students are not able to move directly *x* on point *6* to obtain the solution. They try to move directly the expression on point *27*, and then they make an exploration moving *x*. When they see that the expression *4x+3* is situated on point *27* only if the variable *x* assumes number *6* as values, they seem to understand that they are solving the equation in a completely different way. The insertion of the equation *4x+3=27* and the different color assumed by the corresponding little balls moving the variable on the line, is really effective to construct a justification: "*6 is the solution of the equation, for this reason when x is on point 6 the expression 4x+3 is on point 27 and the little ball here is green... when x is situated on the other values the little ball is red and not green!*" (Martina).

The AL is really important to make visible the relationship between the variable and the equation. The teacher observed that some students, even after the end of this teaching experiment, in spontaneous way, replaced in the equation the value of its solution to see if the solving process was correct.

The proof in AM is constructed on the screen by the teacher supported by students.

<u>Tasks c)</u>

Task c) is a real effective task to understand the different meaning between variable and parameter. Observe that the expression $\frac{2*x+a}{2}-x$ is equivalent to the expression $\frac{a}{2}$.

As consequence, the teacher may always guess the result of the expression because it is not dependent on the value thought by the student.

In AL, the expression $\frac{2*x+a}{2}-x$ does not move dragging the variable *x*. In this way, students can experience that this expression is not dependent on *x*. Students may also observe that the value of this expression is always the half of the value of *a* (which is the value that the teacher requested to add). This explains why the teacher can always know the value thought by the students. After the production of this conjecture, students have to prove it.

$\frac{(2\cdot x+a)}{2}-x$ $(2\cdot x+a)\cdot\frac{1}{2}-x$ $2\cdot x\cdot\frac{1}{2}+a\cdot\frac{1}{2}-x$ $x\cdot 2\cdot\frac{1}{2}+a\cdot\frac{1}{2}-x$ $x\cdot 1+a\cdot\frac{1}{2}-x$ $x+a\cdot\frac{1}{2}-x$ $x+\frac{a}{2}-x$ $\frac{a}{2}+x-x$ $\frac{a}{2}+x+-x$ $\frac{a}{2}+0$ $\frac{a}{2}$	On the left the required proof. The proof in the AM is more complex than in a paper and pen environment. Even if a lot of effort is required by students to prove the equivalence between these two expressions, the system can make visible the rules and procedures of manipulation supporting the comprehension of proof as part of a theoretical system.

<u>Analysis of tasks c) results</u>

When the teacher asks to carry out the following computation, all students are really surprised when she guesses the results of the calculations "*Think a number, double it, add 6, divide the result by two, and subtract from it the number that you thought initially*". The teacher proposes the same task modifying the number to be added to two other students. She asks to add *4* and then *8*.

The students are not able to explain why the teacher is able to guess always the result. The teacher asks to write an expression to translate the computation.

Two pairs of students write the expression $\frac{2*x+a}{2-x}$ but the others write the correct expression.

The teacher asks to insert the correct expression on the AL and move alternatively *a* and *x* to observe the behaviour of the expression.

In the following a part of the discussion developed in the class.

Teacher: What does it happen when you move x?

Alberto: the expression does not move!

Fabrizio: nothing happens!

Teacher: are you surprised?

A lot of students: yesss

They continue to move x

Teacher: and when you move **a**?

Fabrizio: there are some values that are ok...

Giuseppe: but not all values... sometimes the expression disappears

Students speak among them to try to explain why sometimes the expression disappears

Sara: the expression is dependent only on **a**

Teacher: This is an important point! We have seen that if we move x this expression does not move accordingly. This expression does not depend on **x**. On the contrary, we have seen that moving **a** the expression has a particular behaviour. Who can account for this behaviour?

Carlo: wait... if **a** is an even number then I can see the expression, but if **a** is an odd number, the expression disappears! I cannot see it!!?

Dylan: The expression is exactly the half of **a**

Teacher: are you sure that it is always the half of **a**?

Federico: yes it is true, it is the half of **a**

Sara: this is why you guessed the results of the expression... It is always the half of the number you required to add..

A lot of students: yess! It is true!!!

Teacher: but why, in your opinion, the expression disappears when **a** is an odd number?

Carlo: because an odd number divided by *2* is a decimal number

Alberto: it is a decimal number

Carlo: You asked to add only even numbers... otherwise it wouldn't have been possible to divide by two

Teacher: Perfect! In a previous case, in the computational task I asked to add *6*, then *4* and then *8*. They are even numbers. I can calculate the half of an even number. This is why I could guess the result. Is it clear to all?... But what happens if we change the domain?

Danilo: the expression does not disappear

Alberto: it is always the half of **a**, we cannot see it in the Integer numbers because the integers are not decimal numbers!

Teacher: So the expression is always equal to…

All students: the half of a

Teacher: so we can write

The teacher writes on the blackboard $\frac{2*x+a}{2}-x=\frac{a}{2}$

Teacher: can you prove it? Try to prove it in the manipulator.

Only three pairs of students were able to complete the proof by themselves. The intervention of the teacher was required for the other students. However, the constructed proof obliged them to be aware of axioms and rules that are used step by step during the transformation of an expression into another. Only at the end of the teaching experiment students are able to use the AM by themselves effectively.

CONCLUSIONS

In this report we have presented AlNuSet to introduce proof in algebraic domain. The integrated use of AL and AM can be used to make algebraic proof "transparent". In a recent paper (Pedemonte, 2009) it was already shown how the AM of AlNuSet makes proof "visible". In AM the transformation of an expression into another one is not the result of a calculation, but it is carried out by the applications of "explicit" algebraic axioms and rules. Likewise, AL can be used to make "visible" some mathematical concepts (the variable, the parameter, the equation, etc.) that have a crucial role in understanding an algebraic proof. As a consequence, AlNuset maintains the balance between "visible" and "invisible" in the approach to proof in Algebra, where proof in the ordinary educational practice is usually "invisible".

REFERENCES

Arsac, G., Germain, G. & Mante, M. (1991) *Problème ouvert et situation-problème.* Lyon : IREM.

Chiappini, G., Pedemonte, B., Robotti, E. (2008) Using AlNuSet to construct the notions of equivalence and equality in algebra. In M. Kendall and B. Samways (eds), *Learning to live in the knowledge Society,* Springer, pp. 345-349.

Hanna, G., & Jahnke, N. (1993) Proof and application. *Educational Studies in Mathematics, 24*, 421-438.

Healy, L., Hoyles, C. (2000) A study of proof Conceptions in Algebra. *Journal for Research in Mathematics Education*, *31*(4), 396-428.

Mariotti, M. A., Balacheff, N. (2008) Introduction to the special issue on didactical and epistemological perspectives on mathematical proof. *The International Journal on Mathematics education*, *40*, 341-344.

Hemmi, K. (2008) Students' encounter with proof: the condition of transparency. *The International Journal on Mathematics education*, *40*, 413-426.

Boero, P., Garuti, R., Mariotti M. A. (1996) Some dynamic mental processes underlying producing and proving conjectures. *Proceedings of the 20th Conference of the International Group for the Psychology of Mathematics Education PME-XX*, vol. 2, (pp. 121 – 128), Valencia.

Garuti, R.; Boero, P.; Lemut, E. (1998) Cognitive unity of theorems and difficulties of proof. *Proceedings of the International Group for the Psychology of Mathematics Education PME-XXII*, vol. 2, pp. 345-352.

Mariotti, M. A. (2001) Justifying and proving in the Cabri environment. *International Journal of Computer for Mathematical Learning*, Dordrecht: Kluwer.

Pedemonte, B. (2007) How can the relationship between argumentation and proof be analysed? *Educational Studies in Mathematics, 66,* 23-41.

Pedemonte, B. (2008) Argumentation and algebraic proof. *ZDM – The International Journal on Mathematics Education, 40*(3), 385-400

Pedemonte, B. (2009) The Algebraic Manipulator of AlNuSet: a tool to prove *Proceeding of the Sixth Congress of the European Society for Research in Mathematics Education CERME 6,* Lyon, France.

Pedemonte, B., Chiappini G. (2008) ALNUSET: a system for teaching and learning algebra *International Journal of Continuing Engineering Education and Life-Long Learning (IJCEELL)*, Vol. 18, n°5/6, Inderscience Publishers, pp. 627-639.