

DEVELOPMENT OF BEGINNING SKILLS IN PROVING AND PROOF-WRITING BY ELEMENTARY SCHOOL STUDENTS

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ABSTRACT

This paper presents the results of a study regarding the development of deductive reasoning among elementary school students. We have experienced a sequence of 8 teaching and learning lessons in order to develop their primary skills of writing proof in a geometrical context. This sequence, over a 4 months period, was tested with two classes of 26 students aged 11-12 from a single school in Quebec (Canada). Results showed an important increase in most of the students, between the beginning and the end of the sequence, in their ability to reason deductively and validate geometric statements by using theoretical properties rather than measurement.

Keywords: Proof, teaching, mathematics, primary school, geometry.

RESEARCH PROBLEM

It is now a well-established fact in research circles that teaching proof in secondary school is undeniably important. The significance of it may be explained particularly through the many roles that proof-writing plays in the mathematics education of students (Arsac & *al.*, 1992; Duval, 1990; Houdebine, 1990). Among other things, writing proofs, being explanatory by nature, fosters students' comprehension (Hanna, 1995) and helps them develop deductive reasoning (Reid, 1995), critical thinking as well as an ability to support lines of argumentation (Houdebine, 1990).

Key to children's education, proof-writing is also one of the most complex activities and one which secondary students experience the most difficulty with (Houdebine, 1990 ; senk, 1985). The difficulties encountered and the reasons for them are manifold. Amongst the problems observed, two seem to preoccupy experts in current mathematics education research.

The first one relates to the difficulty for students to fully understand the fundamental structure within deductive proof reasoning (Duval, 1991 ; Tanguay, 2005). When writing proofs, students often err in sequencing the inferences constitutive of their demonstration. All they see in it is a discourse, a line of argumentation where propositions are simply added and organized according to relevance only.

The second problem, in connection with the first one, lies in the way secondary school students perceive and use the representations of geometric shapes to write proofs. At the elementary school level, geometry is deemed *practical* (Perrin-Glorian, 2003), as it is linked to spatial sense, visual perception, and shape-building activities. Validating formal geometry statements implies empirical work on figures, whereas in secondary school it relies on theory and very specific axiomatic systems. Thus

redefined, and as Muller (1994) mentions it, shapes constrain secondary school students to reason through concepts rather than drawings; and this is an obstacle for them. Indeed, this alternative justification based on a deductive approach that excludes any conclusion drawn from measurements and observations of geometric figures, seems to generate a lot of difficulties for students. Research in various countries, including Canada, the United States, and France, supports that claim (Chazan, 1993 ; Muller, 1994 ; Paul, 1997). As for Balacheff (1987), he assessed that the switch from *practical* geometry to more *theoretical* geometry caused a breach in the didactic contract agreed upon by both teachers and students, which, he adds, is the main source of learning difficulties when students are introduced to proof-writing. This problem has raised an important question among many researchers: How can students be taught that a practical validating approach based on empirical observations may no longer be reliable when the time comes to write deductive proofs?

THEORETICAL FRAMEWORK

For Perrin-Glorian (2003), solving the problem could imply taking action before proof-writing is being taught in secondary school. She regards the end of the elementary curriculum and the beginning of the secondary cycle as the right time of transition from practical to theoretical geometry. According to her, carefully selected situations could facilitate the switch from one kind of geometry to the other. However, she observes that there is no consensus amongst researchers on how to include the relation between practical and theoretical geometry in the elementary curriculum. We consider that her recommendation seems worth looking into for the following two reasons:

1 – When students are introduced to proof-writing in secondary school, not only are they faced with a new and more rigorous validating procedure like proving, as well as with a new axiomatic system, but they must also learn deductive reasoning in a context of theoretical geometry. And yet, this context of proof-writing is already a problem in itself for students, for acquiring the previously-mentioned elements may not ideally prepare students for deductive reasoning nor to the switch to theoretical geometry. Rather, students should be learning deductive reasoning in more familiar everyday situations the way they do with geometric shapes in validation situations.

2 – Introducing deductive reasoning and more theoretical geometry in elementary school would reduce the breach in the didactic contract described by Balacheff (1987), which is the main source of difficulties for students when first coming to grips with proving. In addition to reducing this breach, a more gradual approach to those elements certainly allows for greater continuity between elementary and secondary school curricula in mathematics.

From practical to theoretical geometry: construction of abstraction

According to Parzysz (1991), geometry activates two types of space: physical space (surrounding space and concrete objects) and abstract space (idealized object). The

first is associated with a more practical geometry while the second is developed through a more theoretical geometry. Objects in a physical space are concrete and they can be observed by the senses, while objects defined in an abstract space exist only in theory or in the form of ideas: they are a mental construct (figures whose existence is ensured by statements as definitions, properties or characteristics). So the transition from a practical geometry to a more theoretical geometry implies that students develop a degree of abstraction. Indeed, a close link exists between understanding of these properties and the use of mathematical abstraction, as pointed out by several authors.

According to Rosh (1978) abstraction may focus on the *properties* of perceived objects. Piaget, for his part, believes that abstraction can take several forms. He distinguishes in fact between construction of meaning through *empirical abstraction* (focusing on objects and their properties) and *pseudo-empirical abstraction* (focusing on actions on objects and the properties of the actions). He associated to these two kind of abstraction the idea of *reflective abstraction* which occurs through mental actions on mental concepts (Piaget, 1972, p. 70). Reflective abstraction is then seen as an activity applied to mental entities rather than physical objects. For Gray and Tall (2007), who also discuss the idea of abstraction, mathematical concepts are the result of a process of abstraction that takes place on a given situation. This abstraction may take the form of « a mental image of a perceived *object* (such as a triangle), a mental *process* becoming a concept (such as counting becoming number) and a formal system (such as a permutation group) based on its properties, with the concept constructed by logical deduction » (2007, p. 23).

If we accept that one of the main purpose of teaching geometry in primary school is to gradually bring students from a physical space to a more abstract space based on the properties of objects, then, we have to consider the importance of the development of abstraction in students and thus, the internalization of properties of mathematical objects and operations on them. This guidance also highlights the importance of taking into account the existence of different forms of geometric perspective, as called by Houdement and Kuzniak (2006): geometric paradigm.

Geometrical paradigms

Houdement and Kuzniak (2006) have defined three geometrical paradigms through which thinking patterns develop differently. Each paradigm is defined by the following components: objects, methods and problems, and therefore each is strongly link to a didactical contract (Houdement, 2007). Each of these paradigms reflects a sophisticated form of geometry and can thus define a specific geometric framework Kuzniak (2006, p.170). It should be noted that the three paradigms are not hierarchical in the sense that they all allow to solve geometry problems properly and efficiently. One is not better than the other, their use will depend on the context.

- *Geometry I* or *Natural Geometry* is the first paradigm whose validation originates in the real, sensible world, according to Houdement and Kuzniak (2006), and where

drawing plays a central role in validation. In this context, deduction is exercised primarily through the perception and the manipulation of objects.

- *Geometry II* or *Natural Axiomatic Geometry* is the second paradigm whose link to reality is not as strong as it is for the first one. It actually aims at understanding reality through axioms, and with the help of which tangible problems may be solved. However, axiomatization is not formal since syntax is not cut off from semantics, the latter referring to reality. In this regard, the source for validation is no longer the sensible, but indeed an hypothetico-deductive process in which intuition and experience still play a role, but to a lesser extent than deductive reasoning does.

- The *Geometry III* or *Formalist Axiomatic Geometry* paradigm is very different from the two previous ones, insofar as it is disconnected from reality. The source for validation is based on logical reasoning only, and not on the sensible world or perceptions.

Deductive reasoning in elementary school

Coppe, Dorier, and Moreau (2005) hold that in order for teachers to demonstrate that proving is meaningful and useful, they have to “force on students the transition to deductive reasoning” (p. 35). Indeed, elementary students are more likely to develop a proper ability for it if they can enjoy teacher’s support, even if that reasoning mode remains an integral part of the human procedural system (English, 1997). For English, textbooks and elementary school curricula must take reasoning processes into account and include informal deduction problems. Furthermore, current research shows that deductive reasoning could be within elementary students’ grasp (Braine and O’Brien, 1991; Daniel, 2005; English, 1997).

RESEARCH GOALS

With this research, we wish to encourage in students a more gradual approach to acquiring preparatory abilities when it comes to proof-writing. We believe that these abilities should be taught as early as elementary school in order to minimize the previously-mentioned breach-related problem. In this light we intend to:

- 1 – Encourage deductive reasoning in elementary students when dealing with mathematical situations.
- 2 – Building from their ability to reason deductively, encourage students to transition from practical geometry (Geometry I) to more theoretical geometry (Geometry II); also, gradually bring them to become aware of the limits of empirical validation for geometric figures and realize how effective more deductive reasoning (Geometry II)-based theoretical geometry can be.

We expect the results of our research to be manifold: For one, it should facilitate the creation of activities for the elementary mathematics curriculum in order to later foster proof-writing skills in secondary students. In fact, no other study has ever looked closely into including basic proof-writing skills acquisition in the elementary

curriculum so far. Finally, our research is in line with the general views of the MELS aimed at encouraging in students the gradual and ongoing acquisition of mathematical skills and knowledge between elementary and secondary school.

Hypothesis

We postulate that if we altered students' relation to figures as well as the role played by properties in the validation process, we would observe in students a spontaneous form of deductive reasoning. Indeed, the change of status for figures could induce new validation processes, no longer based on perceptions or measurements, but on theoretical properties which may only be resorted to through some sort of deductive reasoning.

METHODOLOGY

To meet our objectives, we have opted for a design-based research methodology (Edelson, 2002). This method is cyclic in nature and each cycle consists of 5 stages: 1) Teaching sequence writing; 2) In-class testing of the sequence; 3) Retrospective analysis of experimental data; 4) In light of this analysis, reassessment of theoretical hypotheses, didactic choices, and anticipated learning paths; 5) As a result of reassessment, adjustments are made in the design of the teaching sequence and a new cycle may begin. Also, we have created activities based on works by Coppe and al. (2005), Perrin-Glorian (2003), and Houdement and Kuzniak (2006).

Participants

Two Montreal elementary classes of 25 sixth-graders (11-12 y.o.) were selected for this study prior to which none of the students had ever validated geometric propositions through deductive argumentation. When doing geometry activities, they would only validate geometric situations using measuring instruments.

Tasks

For the first stage of the design-based research methodology, we designed eight tasks that would elicit the spontaneous emergence of deductive reasoning in sixth-graders, as well as spur the transition from practical geometry (Geometry I) to theoretical geometry (Geometry II). The tasks focused on the essential knowledge as identified in the Quebec elementary education program literature, so that they would be easily merged into regular teaching planning and would not add new subject content.

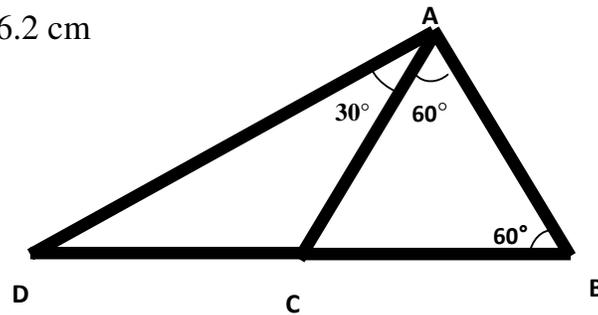
Observation and measurement-based argumentation doubting procedure

The tasks were designed to cast doubt in students on their level of certitude they might reach observing and measuring figures and show them the limits of an argumentation based on that procedure. To this end, we first proceeded to use geometric shapes that made accurate measuring difficult, that is either drawn with exact measures yet in thick and bold lines, or free-hand thus yielding very approximate results. Students could use any chosen method to do each task. Then, they had to answer questions designed to have them process their results which they

also had to compare with those of other peers. As an example, here are two questions asked to students:

Activity 1

Length of side AB = 6.2 cm

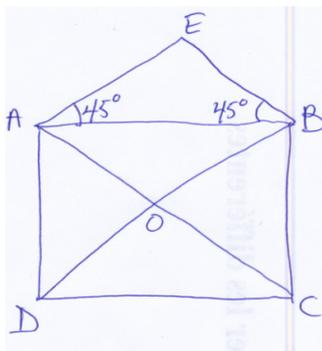


1. a) Measure angle D and segment CD. Write down how you proceed and explain how you reason.
- b) Proceed differently to make up for missing measurements. Write down how you proceed and explain how you reason.
- c) Is there a discrepancy between results 1.a) and 1.b)? If yes, explain why.
2. Compare your results with those of another team.
 - a) Have you got the same results as the other team? If not, explain why.
 - b) Did you proceed as the other team did? If you proceeded differently, indicate which is the more appropriate procedure.

Activity 2

A student draws free-hand the figure below where ABCD is a square whose diagonals AC and BD intersect at O. Then, he draws a triangle ABE on top of the square. Finally, he claims that the quadrilateral AEBO is a square.

Note: The diagonals of a square always intersect in the middle and are perpendicular.



Is the student right? Explain your answer.

In the second step of the methodology (In-class testing of the sequence), we scheduled our experimentation to take place over a period of two months, one hour

per week. We filmed the activities in progress so that we might later analyze how they were performed and how students reacted to them. Finally, students worked in pairs which aimed at prompting debate on validation procedure.

For the retrospective analysis of experimental data (methodological step 3), we observed the students' reactions to the tasks. We also analyzed their understanding of these tasks by using paper trail on written questionnaires, video of two teams in the tasks and the analysis of a logbook of our observations during the experiments. Our attention was also focused to the type of procedure used during the implementation of the tasks (use of measuring equipment or use of theoretical properties).

In the fourth step of our methodology, our objective was to evaluate both the type of geometric paradigm in which students could be located but also to assess how the task favored in the student the shift from paradigm 1 to paradigm 2. To support our analysis, we examined how students used the geometric figure which was provided to support their reasoning. For example, is that students based their answer and justification on intuition, observations or visual estimation, measures or geometric properties and deduction. The results of these tests have allowed us to edit questions, delete or add others to always promote the passage of natural geometry to a natural axiomatic geometry and thereby promote the use of geometric properties and use of deductive reasoning (methodological step 5). Since this experiment took place over a period of three years, we had the opportunity to repeat this sequence of activities on two other occasions with different student groups and thus make each time, adjustments to our work.

PARTIAL RESULTS

At first, more than three-quarters of the students (40 students out of 50) used measurement as the only method to work out missing data for this kind of problem (pré test where they have to find missing data on a geometrical figure with the strategy of their choice). Some used mixed strategies (6 students) whereas two teams (4 students) only spontaneously resorted to a deductive approach using theoretical properties. What we mean by mixed strategies is a crossover approach that uses measurement and theoretical properties. For instance for Activity 1 above, some students used theoretical properties in simple contexts like working out the measure of the 60 degree angle inside the triangle (they used the property of the sum of angles inside any triangle), yet they turned to measurement or visual perception for the more complex areas of the situation, such as working out the length of segment CD. Most students noted that triangle ACD was isosceles, relying on their perception, then concluded that segment AC was the same length as segment AB, that is 6.2 cm.

After the first four sessions, we noticed that students had clearly improved upon their justification procedures. In simple situations where they had to work out the missing measure of an angle in a triangle or a quadrilateral, 42 students out of 50 spontaneously turned to theoretical properties using a deductive approach. In more

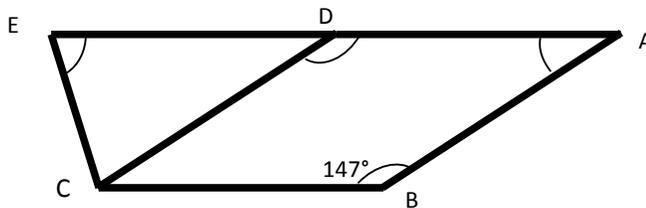
complex situations, we also observed some improvement when more than 50% of the students used a deductive approach appropriately. This improvement is partly due to the constraints imposed by the situations in which students were getting inaccurate results with measurements, thus forcing them to resort to mathematical properties and deduction to find the missing data. We also led the students to compare results between them and tried to explain the differences from one team to another. The students' comments were in line with the imprecision of the measure (which in our case was strengthened by the bold line that accentuates the vagueness on measure or the hand drawing figures).

At the end of our experimentation, all students were able to identify spontaneously the limits and lack of precision of a measurement and observation-based approach, as well as call on theoretical properties to validate simple geometry statements (problems where only one data is missing in a simple geometric figures). ver, this result does not suggest that all students have acquired the ability to produce simple proofs or arguments based on properties using deductive reasoning appropriately. Besides, more complex situations remained difficult for some students leading them to use mixed strategies (measurement and deduction); and so did situations requiring validation, and where data were not provided. For example, in order to solve the problem below, students were able to use measurement as well as theoretical properties, which they knew well by then. However, the absence of numerical data significantly hampered task completion and caused students' strategies to revert to measurement and observation.

Let ABCD be a rhombus and let D be the midpoint of segment AE.

We know that the sum of the four angles of a rhombus is 360 degrees.

1. a) Find the measure of angles A, B, and E.



DISCUSSION

The eight sessions that we had scheduled over a period of two months allowed us to bring most students to switch from practical (G1) to theoretical geometry (G2) spontaneously. In doing so, we had set for students an environment conducive to the use of deductive reasoning in validation situations. And yet, our experimentation validated our initial hypothesis, that is deductive reasoning shows to be a demanding process that requires time and extensive experience to be exercised properly. In fact, it had taken four sessions before students started showing some improvement in reasoning deductively. Yet, the students who improved the least were still able to

grasp the geometry concepts and skills in the tasks given in the course of our experimentation. We believe that this type of activity can be very productive when it comes to teaching and learning the geometric properties of given figures; it could also apply with most geometric concepts covered in class during the school year.

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