UPPER SECONDARY SCHOOL TEACHERS' VIEWS OF PROOF AND THE RELEVANCE OF PROOF IN TEACHING MATHEMATICS

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This paper reports some results from a case-study on five Swedish upper secondary school teachers' views of proof. We describe the teachers' views of what constitutes proof and focus particularly on their views on the role and relevance of proof in teaching mathematics. We identified two basically different ways of relating to the teaching of proof in our data that consists of interview transcripts and protocols of observations of lessons. We discuss these views from a socio-cultural perspective on learning and Wenger's concepts of community of practice, reification and participation.

Keywords: Mathematical proof, teachers' views, community of practice, reification, participation

INTRODUCTION

The proposed new curriculum for upper secondary school mathematics in Sweden is manifesting a more explicit focus on proof and proving (Swedish National Agency for Education, 2010). The same trend is visible in curricula also in other countries. The guidelines from NCTM already in 2000 recommend that every student from pre-kindergarten through grade 12 experience proof and proving in different ways in their school mathematics (NCTM, 2000). Proof has also recently obtained a stronger status in curricula in Italy (Furinghetti & Morselli, 2010) and Estonia (Hemmi et al., 2010). Common to many of these curriculum reforms is the emphasis on the relevance of proof for *all* students. However, it is not clear to what extent this emphasis is coherent with teachers' views of proofs and the role of proof in teaching mathematics.

The American secondary school teachers in Knuth's qualitative interview study (2002) express the view that *formal* and *less formal* (Knuth's classification) proofs are not something beneficial for all students. Instead the teachers mainly find this kind of proof appropriate for students studying science or other mathematically intensive educations. This view is contrary to the visions put forth in NCTM 2000. *Informal proof* – i.e. explanations or empirically based arguments – is not considered as valid proofs but regarded by the teachers in Knuth's study as a vital part of the education for *all* students. The view that proof – in the traditional sense (c.f. Reid, 2005) – is not something for all students is also present in the teachers' responses in a multinational pilot study carried out in Sweden, Estonia and Finland (Hemmi et al., 2010). Although distinguishing several functions that proof can serve (c.f. de Villiers, 1990; Hanna, 2000; Hemmi, 2006), the teachers mainly consider proofs and

proving relevant for students studying courses that prepare the students for university studies in programs that include mathematics, for example technology and natural science. Both studies actualise the question of what constitutes proof in upper secondary mathematics as well as what kind of proofs and proving activities are appropriate and relevant for different kinds of students.

Research suggests that the way that curricular guidelines are implemented in the classrooms is greatly influenced by the teachers' experience, views and conceptions (Corey & Gamoran, 2006; Remillard, 2005). This underlines the fact that school reforms are never launched into a vacuum. Rather they are introduced into a community of teachers with their own conceptions of proofs and about what role proofs can or should play in teaching mathematics. To understand how reforms are going to be implemented and, on a larger scale, what role proofs play in the teaching of mathematics, it is important to understand teachers' conceptions of proof and proving. This is not the least true in Sweden, where the national curriculum leaves substantial space for local interpretation and appliances.

In this paper we report parts of the results of a case-study about five Swedish upper secondary school teachers' conceptions of proof (Reuterswärd, 2008). We discuss the teachers' views on what constitutes proof in upper secondary school mathematics and analyse the ways in which the teachers talk about the role and relevance of proof in their teaching.

We focus in particular on the following questions:

- What constitutes proof according to upper secondary school teachers?
- What are upper secondary school teachers' views on the role and relevance of proof in teaching mathematics?

Theoretical stances

The analysis is based on a socio-cultural perspective where learning is considered to take place through active participation in a *community of practice*. Proof is according to this perspective seen as an artifact that can mediate mathematical knowledge. Wenger's (1998) concepts *negotiation of meaning*, *reification* and *participation* are used as central tools of analysis to understand and discuss teachers' talk about proof (c.f. Hemmi, 2006; Reuterswärd, 2008).

By a *community of practice* we mean people who share experiences and have a common goal or purpose of some kind. Both a group of teachers at an upper secondary school as well as a group of students studying mathematics fall under this definition. Every community of practice has its artifacts, or *reifications;* tools like a calculator, a mathematical formula or a proof, mediating mathematical knowledge. They create focus points around which teaching can be organized. But the symbol for pi (π) or the proof for the Pythagorean Theorem do not carry any meaning of their own. Without people *participating* in *negotiating* their meaning, they are mute,

meaningless (c.f. Wenger, 1998, p 52-54). It is when people engage in a mathematical practice and participate in negotiating the meaning of proofs that they can be meaningful and convince, explain and communicate mathematics.

The meaning of reifications must be negotiated within the community of practice. Hence, proofs taken from a mathematical practice must be renegotiated, within upper secondary school. It is therefore not self-evident that proofs will play the same role in school as they do within the mathematical science. Pushing to extremes, the role of proof needs to be renegotiated within every single classroom. The aim of this paper is to shed light on how teachers renegotiate the meaning and role of proof within an upper secondary school context.

Method

The paper is based on a case-study aimed to describe what constitutes proof in upper secondary mathematics from a teachers' perspective (Reuterswärd, 2008). The case-study had a phenomenographic approach and hence sought to qualitatively describe teachers' views and conceptions, believing that a concept can only be perceived in a number of qualitatively different ways. The study was conducted at one ordinary Swedish upper secondary school that includes the Social- and Natural Science Programs. Five teachers were chosen to participate in the study. They were of different ages, both genders, with different teaching background and at the moment of the data gathering teaching different groups of students. Such a heterogeneous selection of participants can represent qualitatively different views (Marton & Booth, 1997).

The data was collected using semi-structured interviews. In order to enhance the richness of the data the teachers were given an outline of what themes were going to be discussed during the interview. Hence, they had the chance to reflect on the questions beforehand. Several days before the interviews they also received a questionnaire with statements chosen from a questionnaire that was piloted by Hemmi during the time of the data gathering. By choosing a number 1-4 the teachers stated to what degree they agreed with the statements. Thus, we had written material to come back to during the interviews and thereby inconsistencies or uncertainties could be resolved. In the beginning of every interview the teachers were asked to describe how they defined a mathematical proof. The interview focused on the following themes: how the teachers perceived student attitudes toward proofs, how the teachers worked with proof and proving in their teaching and how they viewed proof in general. Every interview was tape recorded and transcribed. Unstructured observations of lessons, at least two with every teacher, were also used to further triangulate the method and validate the results.

RESULTS

Teachers' views of proof

There is a broad consensus among the teachers about what constitutes mathematical proof. Proof is something specific for mathematics that from certain premises, through logical argument, step by step, deduces the truth of a statement. Some of the teachers stress that the premises should be axioms or theorems that have previously been proved. The characteristics of proof are a mathematical language, an unassailable logic and a specific structure. The teachers hereby distinguish between proof and other types of arguments. Proofs for them are more formal:

> Teacher: It [a proof] is that one, from certain premises, with given presuppositions, can strictly, step by step so to say, that is totally true, arrive at something that has to be general.
>
> Teacher: If someone says proof you feel that it is a bit heavier piece, so to say.
>
> Teacher: It follows that, hence, implies, is equivalent with. It's a bit of those things. You use some of those words that are associated with proofs. That's when I think it's a real proof.

Emphasizing that proof through logical derivation, structure and formal language deduces the truth of a statement, these teachers can be said to embrace a *traditional concept of proof* (c.f. Reid, 2005). This view of proof was also represented in Knuth's (2002) study by American teachers, referred to as *formal* or *less formal* proof.

Although the teachers see the formal mathematical language, structure and logic as distinguishing proofs from other types of arguments, there is some uncertainty as to where to draw the line. Tasks like "show that left hand side equals the right hand side" and using general methods to solve problems are examples of areas that the teachers mean "*are in the vicinity of proof*" but for which the term "proof" is not used. This view is coherent with the view that Knuth calls *less formal* proof: a general argument that lacks a rigorous mathematical structure.

One teacher also gives a slightly more informal definition of what proof can be in upper secondary school:

> Teacher: Proof in the sense that they [the students] should understand that it's not arbitrary, then you do it most every day. To make them understand that mathematics is a logically built system, and it's not a coincidence that we have to do things in a certain way.

This can be compared to what Knuth calls *informal proof* i.e. justifying by explanations or examples. There is, however, no doubt that examples and empirical investigations are not considered valid proof, neither in Knuth's nor in Reuterswärd's study, and the concept informal *proof* is therefore somewhat unfortunate.

The difficulty to draw the line to what counts as proof is not a problem reserved to school. Instead it is a forever ongoing discussion for which different answers are given within separate mathematical disciplines and historical contexts (Hanna & Jahnke, 1993). In Wenger's (1998) terms we can say that the meaning of an artifact has to be negotiated within every practice.

The role of proof in the classroom

Common to all the teachers in this study is that they all think of proof as central to mathematical thinking, and that they distinguish several functions that proof can serve in teaching mathematics. They see proving as a desirable competence and as a suitable challenge for the well achieving students. They also agree that the role of proof in the classroom must be determined in relation to every single student group. Their views on how this valuation should be made, especially considering the low achieving students or students in the Social Science Program, are however different. Two separate views were distinguished in the study. This should not be understood in a way that the teachers can be said to solely support one view or the other. On the contrary, the teachers' ambivalence might be the most prominent feature of their views on the role of proof in teaching mathematics.

One view embraces the thesis that *proof is something for all students* and that it is important that the teacher takes the step from examples and informal reasoning to proof. The explanatory function of proof, in particular, motivates that proof is something for all students, even the low achieving ones.

> Teacher: I don't really think that you should assert anything to students without in some way proving it. If it is sometimes easier to make a geometric proof or a purely theoretical proof… /…/ I definitely don't think you should just state it: now you do like this. Why should you do like that?

> Teacher: With the students I've got I know that there are many who think that, if there is anything you should leave out, it's proof. And then I have to say that I don't quite agree because that could be what gets them back on track.

The underlying idea here is that understanding why something is the way it is in mathematics, is a vital condition for the students to experience mathematics as meaningful, and to be able to use it in the right way. As a teacher you therefore run the risk of losing students if you don't take the step to more formal arguments. The basic assumption is that students can and want to understand.

Furthermore, according to this view, proofs don't have to be difficult. There are abstract and concrete, easier and more difficult proofs, and according to this view it's up to the teacher to present them in a way that can appeal to every student.

The other view is characterized by the idea that *proof is not necessarily something for all students*. The teachers share the basic positive attitude towards proof, ("*I am pro proof, I really am.*") but mean that the students lack the necessary qualities to

realize the meaning potential of proof in the classroom. The students lack sufficient mathematical knowledge or sufficient experience of proof:

> Teacher: If you can hardly calculate the area of a triangle it's really hard to explain what it [proof] is.
>
> Teacher: It will be really difficult because they've hardly done it at all before. They'll only scream when I do it.
>
> Teacher: If they don't accept the proof it's because they have gaps in their [mathematical] knowledge. Then it's really hard to accept proofs.

According to this view, the students' insufficient previous knowledge makes it impossible for them to participate in negotiating the meaning of proof. The teachers mention, for example, that the students need to know the quadratic rule in order to understand the proof of the quadratic formula, and master similarity to understand one of the most common proofs of the Pythagorean Theorem. As a teacher you therefore run the risk of losing students if you conduct general or algebraic reasoning.

The basic assumption here is that proof generally is something advanced that only a few students have the ability to master. According to the teachers some students are also not interested in proof; they are content with examples and only want to know 'what to do'. Proof is not necessarily something that is relevant for all students, in many cases it's enough to explain through examples:

> Teacher: These Social Science students for example, I don't know if they have any use of it later, when they graduate, because it's probably not too many of them who'll choose a mathematical education.

Reification, participation and negotiation of meaning

Some of the teachers' statements express that understanding mathematics makes the students more capable of using it correctly, more inclined to thinking math is fun and more apt to remembering what they have learned. In Wenger's terms: understanding mathematics enhances the students' inclination and ability to *participate* in the community of practice of mathematics as it is exercised in the classroom.

This is the leading idea in the view that we have called *proof is something for all students.* According to this view, understanding why something is the way it is in mathematics is a requirement if the students are to experience mathematics as meaningful and to be able to use it in the right way. As a teacher you therefore run the risk of losing students if you do not take the step towards such general arguments as proofs. In other words: Without *reifications* like proofs, the students may experience mathematics as meaningless or hard to understand. The reification offers a necessary structure, an abstraction to tie the knowledge to. The other view, which we have called *proof is not necessarily something for all students***,** means instead that some students lack both the motivation and the knowledge to be able to participate in

negotiating the meaning of proof. According to this line of thought, one runs the risk of losing the students if one *does* take the step towards such general arguments as proofs. Thus, the reasoning is turned over, and it is claimed that without possibility to participate – without previous knowledge or motivation – the reification is meaningless anyway. In line with this idea, Hanna and Jahnke (1993, p. 434) point out that the students need well-founded previous knowledge to be able to negotiate the meaning of proofs.

We can see that both views express the same main goal; to make the students understand mathematics. The different views can be described as different answers to the question of what proportion between reification and participation that is the most meaningful to the students. This can be compared to Wenger's (1998, p. 65) remark that the proportion between reification and participation always has to be negotiated, and that different proportions lead to different possibilities to create meaning. An abundance of mathematical artifacts such as symbols, formulas or proofs do not create any meaning without the possibility of participation. But in the same way participating without reifications can seem meaningless. The sole use of examples, empirical investigations and lots of time can be experienced as useless without mathematical reifications to focus the practice on; without concepts, symbols or formulas to structure the mathematical knowledge. It is in the tension between reifications and participation that meaning can be created in the community of practice.

All the teachers in this study express that proof and general arguments are something desirable. But like we have described above, some of them state that empirical investigations or examples can replace proof in groups where the students are not susceptible to formal reasoning. Even those teachers who in principle mean that proof is something for all students, point out that using informal methods is a good first step towards proof. Informal methods can include letting the students see examples, investigating certain cases (empirical investigation) and trying to find patterns. We can interpret these methods as strategies to increase the students' motivation and ability to take active part and engage in understanding reifications like proof.

Although the teachers express understanding as the main goal of their teaching, it is not self-evident that all students identify with the community's goal to understand. Above all, not all of them are interested in proof. *"Some students are only like "what to do?"* as one of the teachers in this study put it. We can interpret it as if the students simply are not interested in identifying with the goal of the community of practice. Maybe they want to practice law and do not see any use in specifically mathematical reifications like proof. It is also possible that they want to understand, but that they through years of failure have developed an identity of non-participation (Wenger, 1998, p. 165) – 'I don't know mathematics'. That means that they have

developed an identity where they do not see themselves as participants in the community of practice.

The concept *community of practice* can also help us to see that the teachers have several different practices to consider when they value the role of proof in their teaching. Some teachers point out, for example, that one reason not to deal with more proofs is that "the students haven't experienced them before, in compulsory school." This is in turn probably related to the fact that proof is not at all mentioned in the Swedish steering documents for compulsory school (Swedish National Agency for Education, 2000). But also future practices play a role in the teachers' views. Upper secondary school is meant to prepare for higher education and future professions. Which these are likely to be, matter to some teachers.

DISCUSSION

In this paper we have reported parts of the results of a case study of five Swedish upper secondary school teachers' views of proof (Reuterswärd, 2008). We have particularly focused on the teachers' conceptions of proof and the role and relevance of proofs in the context of upper secondary school mathematics. In doing so we have presented two different views on the role of proof in upper secondary mathematics that we've called *proof is something for all students* and *proof is not necessarily something for all students.*

In the beginning of the paper we drew attention to the fact that many curricula around the world manifest a revaluation of the role of proofs in teaching mathematics. Common to the guidelines in the NCTM (2000), Italy (2003) and the proposed new curriculum for upper secondary school in Sweden (2010), is the emphasis on the relevance of proofs for all students. This rhyme well with the view that we have called: *proof is something for all students.* But how will these reforms turn out in a community of teachers where one of the views is that *proof is not necessarily something for all students*? This question is not the least relevant considering that this view is expressed not only by the teachers in this study but also by American teachers (Knuth, 2002) and other Swedish teachers (Hemmi et.al, 2010). These results raise the need for these new curricula to be anchored in the community of teachers supposed to realize these visions.

We can only speculate as to what has been the driving force behind more focus on proof and proving in the curricula. We find it likely that it has something to do with the extensive functions – such as explanation, verification, communication and transfer (c.f. de Villiers, 1990, Hanna, 2000, Hemmi, 2006) – that proofs can serve in teaching mathematics. It is therefore important to notice that the teachers in Knuth's (2002) study seem to regard proofs mostly as a topic to study, rather than using them to teach mathematics. This view is not shared by the Swedish teachers in this study. They chiefly see proof as a tool that can mediate mathematics. This is probably partly due to the fact that proof is hardly mentioned in the present mathematics curriculum

in Sweden. Yet, one can ask whether a sudden focus on proofs in the new curriculum has the effect that the teachers start viewing them first and foremost as a topic to teach, rather than a gateway to mathematics. Yet, both approaches are needed in the teaching of mathematics (c.f. Hemmi, 2008). This highlights once again the need for the curriculum reforms to be anchored in the community of teachers. This can be done by making the teachers aware of the arguments behind the new guidelines, and offering them the tools to introduce proofs also in groups of students lacking motivation or prior knowledge.

This being said one can wonder what the implications are if proofs become part of the mathematics education for all students. It is well documented that many students find it difficult to understand, conduct and value proofs (e.g. Healy & Hoyles, 1998; Selden & Selden, 2003). The teacher in this study who says "*If you can't calculate the area of a triangle, it's really hard to understand what a proof is.*" expresses a point not to be taken lightly. Introducing proofs to these students could possibly further their identity of nonparticipation in the mathematics classroom (c.f. Wenger, 1998). On the other hand, experiencing the derivation of the area formula for a triangle could help students to find proving as meaningful. One way to proceed might be to gradually formalize the use of justifications throughout the curriculum as is the case in the American guidelines from NCTM (2000). However, it certainly calls for research studies about how these reforms are being implemented and what effects the focus on proofs has in different student groups. It also underlines the need for research exploring different ways of teaching proofs to different kinds of students.

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