THE ROLE OF LOGIC IN TEACHING, LEARNING AND ANALYZING PROOF

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This paper addresses the importance of forms and usage of logico-mathematical reasoning in teaching, learning and analyzing proof and proving and their relevance to understanding students' reasoning processes. The question is considered of whether students in upper secondary schools can improve their reasoning and proof abilities by taking an introductory course in logic. To that end, a particular conception of proof is presented and its relation to argumentation, which shapes the whole didactic situation entangled in Greek secondary education. Some preliminary results are analyzed which show that the course mainly succeeds in strengthening students' inference abilities and offering them problem-solving methods to use in other courses as well.

Keywords: proof, argumentation, quantification, conditional inference, teaching logic

INTRODUCTION

A plethora of research papers on the teaching and learning of proof and its relation to other forms of justification and argumentation was produced recently giving rise to fruitful discussions and debates. Our study focuses on the importance of the logic involved in proving processes in order to ease students' performance in proof and proving. Researchers have identified numerous difficulties in formal reasoning and proof students face both in high school and at university (first years). In a study of first-year university students' understanding of logical implication Durand-Guerrier (2003) emphasised that students' difficulties are due to the complexity of this notion and the implicit use of quantifiers in high-school mathematics classrooms which prevents the emergence of contingent statements in open statements like '$p(x) \rightarrow q(x)$'. Even advanced students encounter problems in negating a given implication or a quantified statement (Durand-Guerrier,2006). Similarly, Küchemann & Hoyles (2002) showed that only the minority of high-attaining students in English schools stated that a conditional statement and its converse are equivalent and used deductive methods as warrants for their conclusions whereas the majority of students opted for empirical evidence to justify their conclusions. Tall (1989) reported A-level-pupils' problems in distinguishing between the statements 'if P then Q' and 'if Q then P' and Tall (2008) proposed to tackle students' difficulties in handling multiply-quantified statements via a 3-mental-world model (*embodied, symbolic, formal*) in mathematics. The following studies have also documented ways in which students' inability to deal with logic behind the proof and to disentangle logical elements from the content of mathematical statements deprives them from understanding and constructing proofs. Thus Selden & Selden (1995) provide evidence of students' inability to unpack informally written mathematical statements into the predicate calculus language and reliably relate them with the top-level logical structure of their proofs. They identify as a major source of students' difficulties in discerning the logical structure of theorems the lack of understanding the meaning of quantifiers. Epp(2003) identified as a main source for students' problems with formal mathematical reasoning the differences between formal and informal discourse related to if-then and quantified statements. The logical distinction of a conditional statement and its converse and negations of both quantified and if-then statements as well as students' previous inadequate instruction cause students many difficulties. Some teachers' practices of omitting proofs of theorems, relying only on empirical justification and de-emphasizing general principles and formal proof (due to limited time and/or curriculum pressures) may lead to even fewer numbers of students leaving secondary schools without a real sense for deductive argument.

One agrees then with Durand-Guerrier's (2005) claim that quantification provides a powerful tool used in mathematics education to analyse proofs and to understand students' difficulties with proofs since 'logical validity is a prominent criterion for analysing proofs in a didactic perspective, prior from the focus on proving to convince or proving to explain'. The same claim is advanced by Dubinsky, Elterman & Gong (1988): quantification is of outmost importance for developing a sophisticated understanding of calculus notions such as limit and continuity; yet it is 'one of the least often acquired and most rarely understood concepts at all levels, from secondary school on up even, in many cases, into graduate school'. They describe an epistemological quantification schema along Piagetian lines, a *genetic decomposition of quantification*, according to which students should make the mental constructions of coordination, generalization, interiorization and encapsulation in order to handle adequately single/multiple quantifiers and negation of quantified statements. Dubinsky & Yiparaki's studies (1996, 2000) provide evidence showing that students have both syntax and semantics difficulties, that is, problems with both interpretations and the logical structure of statements. For example, a significant fraction of students interpreted an **EA** statement $(\exists x)(\forall y)P(x,y)$ as the same as an **AE** statement $(\forall s)(\exists t)\,R(s,t)$. Transfer problems were also identified from real-life situations to mathematical ones. Hence the authors' suggestion to help students not by analogies and explanations from natural language but by remaining in the mathematical realm especially in predicate calculus when dealing with concepts not well understood and for which analogies and mental pictures are hard to locate (1996).

It follows from the above studies that student achievement with respect to implication, quantification and logical structures -indeed invaluable for students' mastering the proving process- is relatively low. Thus the logician's complaint about 'the lamentable neglect of basic logic in our high-school curricula. Our high-school graduates are bereft of the concept of consequence or deducibility..They are not required to have internalized the basic rules of inference of modern logic… Mathematical proof is the paradigm of logically rigorous reasoning…But seldom do they learn how to prove, from first principles, why those formulae work..So how could they be expected to understand the crucial notion of the refutability of a scientific theory? Without a grasp of deducibility, they cannot acquire the concepts of consistency and inconsistency. Nor can they attain the notion of logical independence' (Tennant, 2007) entails that we should find ways to make students familiar with the standards of mathematical argumentation. Our claim is that teaching them logic **is** such a way. These issues will be considered mainly from a philosophical perspective related to the nature of proof, logic and argumentation, but also from an educational one related to the ways practicing mathematicians could enhance students' proof abilities.

THEORETICAL FRAMEWORK

If the target in mathematics education is understanding, the question of what counts as proof is crucial, that is, which proofs have a right to play a role in understanding mathematical knowledge. The next two sections approach proof with respect to its relation to argumentation, its nature and conceptions. These issues are informing the didactic choices of the logic course and they are framing the whole design of it.

Proof & Argumentation

Hanna & de Villiers (2007) claim that understanding the relationship between argumentation and mathematical proof is essential for designing learning tasks and curricula aiming at teaching proof and proving. When researchers view mathematical proof as distinct from argumentation the educational implications are very different from those who see argumentation and proof as parts of a continuum. Duval (1991) pointed to a dramatic discontinuity between deduction and argumentation: in the former one must distinguish between status and content of a statement; and status (e.g. hypothesis, conclusion) in contrast to content, depends only on the organization of deduction, and therefore, he claims, the validity of a mathematical statement, unlike in any other field, relies only on logical/operational value, not on epistemic value. From the other stand, Boero, et al (1996) assert that it is only by engaging in conjecturing and argumentation that students develop an understanding of mathematical proof. They claim that a *'cognitive unity'* exists between the production of a conjecture during argumentation and the successful construction of its proof. This conjecturing/argumentation approach to proof received many criticisms since the natural language employed in students' argumentation patterns seems to be in conflict with the logic associated with deductive reasoning. So Balacheff (1991, 1999) claims that argumentation is an obstacle to the teaching of proof. Following Perelman 'whereas mathematical proof in its most perfect form is a series of structures and of forms whose progression cannot be challenged, argumentation has a non-constraining character. It leaves to the author hesitation, doubt, freedom of choice; even when it proposes rational solutions, none is guaranteed to carry the day' (in Balacheff 1999), he argues that the major source for the rupture between argumentation and proof is 'the necessity of the latter to exist relative to an explicit axiom system'.

Barrier et al (2009) claim that their model-theoretic approach calls for continuity between argumentation and proof. In their opinion the proof process is a dialectic one between syntactic and semantic aspects in proof construction. Support for this view comes from Azzouni (2005) who says that traditional mathematical proofs, in contrast to formal derivations, seem to be intrinsically semantic, not syntactic, by recognizing the properties of the objects the inferences are about. He claims that the certainty mathematicians have about their successful ordinary proofs does not come from a deductive justification, but from what he calls '*inference packages',* which explain mathematical intuition, the semantic presence of objects in proofs and tacit reasoning. Understanding a proof means being familiar to an inference package, that is, when the concepts employed in a proof facilitate the use of an inference package.

The Nature of Proof

Should we compare Azzouni's view with Rav's view, we could see striking similarities:

….the usual [mathematical] proof…involves a somewhat large jump from statement to statement based on formal technique and on intuitions about the subject matter at hand (Azzouni, 2005).

…..the usual ('ordinary') proofs produced by mathematicians have subtleties of meaning that go beyond the application of logic but often contain conceptual bridges between parts of the argument rather than explicit logical justification (Rav, 1999)

Azzouni's inference packages also relate to Rav's conception of proof as a general method, as a toolbox of strategies and analytic procedures. Here the criterion of the value of proof is whether the proof offers a new method, thus enlarging our mathematical machinery. Rav' s writings challenged the traditional conception of mathematical proof, that is, the Fregean logicism-theses or what he calls 'Hilbert's Thesis' -every mathematical proof should be convertible into a formal derivation in a suitable formal system. On this traditional view, which emerged out of the foundational concerns at the beginning of the 20th century, rigor and formalizability are taken as necessary features of proof. However, though such a conception offers an essential formal tool to mathematical practice, it is unable to sufficiently model real mathematical proofs and the practice of proving and thus unable to explain the nature and our understanding of mathematical knowledge. What came out is a 'defense' of both the activity of constructing proofs of theorems in axiomatic systems and at the same time the activities which lead to discovering mathematical facts, establishing conjectures, exploring, generalizing and reasoning. Resnik (in Detlefsen, 1992), claims that most historical proofs are informal, unaxiomatized demonstrations, which he calls *working proofs*.

From our theoretical discussion, we inferred a particular conception of proof which many mathematics educators and philosophers regard as crucial in the learning and teaching of proof, one that merges in a mild coexistence formal language and natural language. When teachers teach proof and logic, proof comprehension, identifying proof's important components and the proper logic for it are all of practical significance. Balacheff (2008) says our epistemology of proof, that is, our theory on truth and validity, clearly defines our theoretical and research framework, even before we choose a *problématique*; it is the first 'deadlock' we have to consider. Our own instructional decisions were informed by the above epistemological conception of proof and logic, in which both formal and intuitive aspects of mathematical proof coexist.

DESCRIPTION OF THE LOGIC COURSE

An introductory course in logic is offered by both authors in an experimental secondary school in Athens, Greece, where such a course is not usually offered by secondary schools. The course *'Logic: Theory and Practice'* is being offered to students since the 2003/2004 academic year. It is an optional course for 17-18 years old students in the secondary-tertiary transition. The course is given during the whole academic year twice a week for students both from mathematically-orientated classes (Group I) and from theoretically-orientated classes (usually with a good background in mathematics)(Group II). Groups may vary from 20-27 students per class. There is a written examination at the end of the year (concerning the total annual grades).

The course aims generally to develop student skills in the propositional and predicate calculi and to encourage students to exercise these skills in applications that arise in both mathematical and other courses and in practical reasoning situations as well. More particularly, some of the aims as described in the textbook (Anapolitanos D., et al, 1999) are:

i. To teach logic as a set of key-concepts and logical rules and structures
ii. To approach logic both as an abstract, autonomous discipline and as a useful tool for other knowledge areas to apply.
iii. To cultivate logical thinking in students and the ability to build lines of reasoning for themselves.
iv. To aid students in formulating their views with clarity and precision and producing convincing arguments to support them.

There are certain constraints we have to bear in mind each year when we design the course. To start with, since students are at their critical period of getting prepared for their entry exams to universities, time pressure and heavy work duties from their part are definitely serious constraints. Second, this is an optional course which students from various mathematical backgrounds choose either from genuine predilection or from sheer curiosity. Though this may be a challenge, bridging such diverse groups of students is far from being an easy task. When Groups I and II are taught separately things seem more straightforward. The course is taught from a mathematical (Group I) or from a (more) rhetorical/practical (Group II) perspective by each author respectively. However, the years during which we teach mixed groups of widely different goals and needs we have to carefully prepare and readjust our teaching materials. The course mainly attracts students who opt for studying mathematics or science in tertiary education as well as students with engineering aspirations and whose interests are mostly in the formal aspects of the course. Students who anticipate careers in law school or political science show a clear interest in correct modes of reasoning as applied to argumentation strategies and the 'rhetoric of conviction'. And, there are also students with no definite career plans who want to get acquainted with correct argumentation techniques for actual reasoning situations.

In any case, employing suitable techniques and finding appropriate tools for teaching logic and reasoning such various groups of students demands special care. We view these constraints as opportunities for creating logico-mathematical tasks and experiences for 'Logic for All'. For students of Group II a gentler introduction to logical issues is attempted. Our material relates to the application of valid forms of inference to argumentation theory and practical reasoning. The emphasis here is on the social dimension of proof and on degrees of persuasion in the communication process rather than on pure formality. Students use such theory to discern argumentative structures from texts and evaluate arguments in their language, philosophy and ethics courses. And many of them get prepared for Debate Classes and participate in Argumentation Competitions (Argument Attack and Refutation) in the Parliament 'against' other schools' competitors.

The Content of the Course

(I) The T**extbook** includes the following units:

1. *History of Logic*: Aristotelian logic & Theory of Syllogisms, Stoic Logic, Medieval Logic, Modern Logic, Philosophy of Mathematics.
2. *Propositional Logic I*: Connectives, Truth-Tables, Translation & Formalization, Tautologies, Contradictory & Contingent Proposition Forms, Logical Equivalence, Valid & Invalid Argument Forms.
   Propositional Logic II: Testing Arguments for Propositional Validity, Formal Proof, Rules of Inference, Natural Deduction, Conditional Proof, *Reductio Ad Absurdum*.
3. *Predicate Logic*: Quantifiers, Scope of Quantifiers, Existential & Universal Quantifiers, Multiple Quantification, Syntax & Semantics.
4. *Elements of Practical Logic*: Argumentation Theory, Arguments, Premises & Conclusions, Types of Arguments, Deduction & Induction.

(II) S**upplementary Material**

A workbook of handouts and lecture notes is produced including exercises with solutions and short assignments. Logical puzzles and paradoxes which are fun and very good in introducing the concept of reasoning are also included. The software package *Tarski's World*, which treats together syntax and semantics in a natural and playful way, is used as a supplement to our logic textbook. We also use Bray's (2006) aspects of soccer games and Gardner's (2001) puzzles in order to translate phrases and sentences from common argumentation into symbolic logic. Fictional cases based upon Agatha Christie's books form another challenge.

Finally, the course is enriched by extracurricular activities such as video sessions on relevant topics and our "Math & Science Club" activities. The clubs are run by joint volunteer secondary-school teachers in which we have the opportunity to exercise in action the *'theoretical physicist'* metaphor and the related proof conception which springs from a mathematico-scientific combination (Jahnke 2005).

EVALUATION OF THE COURSE

Students (in both Groups) who sat the course in previous years made the following observations. The course proved invaluable in arguing logically and correctly evaluating patterns of reasoning as to their validity and soundness during their entry exams. It helped them building bridges among disciplines and digging deeper to discern correct structures. They are practiced to uncover implicit forms of reasoning and their performance improves significantly. Calculus students find the course very helpful, especially when tackling a problem they make the right connection to the logic required, i.e. in quantification, (transfer of logical knowledge) and they experience the 'aha momentum' (relation between the logic taught to the logic used). There are also indications of benefits for students' general intellectual development since clarity of thought and correct forms of reasoning have value far beyond the classroom. The same deductive logic used in careful mathematical reasoning is used to tackle tough problems in other fields.

Preliminary Results

We briefly refer to particular logical themes that our students have found both interesting and challenging over the years. These results guide our focus areas in the course over the years.

1. The distinction between truth and validity

Students get surprised when they learn that sentences like 'The argument is true' or 'The sentence is valid' so commonly used in everyday life are incorrect since validity is a property of arguments whereas truth is a property of individual sentences (category mistake). The first discussions showed that these mistakes are very frequent and sometimes reinforced by the loose use of natural language.

2. Conditionals

Conditionals and the mathematical concept of implication in both argumentation and proof are absolutely necessary for students to grasp various kinds of reasoning. This focal area causes certain amounts of confusion even to successful mathematics students. Of course such students are more prepared to handle the notion of implication. Deloustal-Jorrand (2004) reports that for having a good understanding of implication it is necessary to know and establish links among its various conceptions from set theory, formal logic and deductive reasoning-areas with which high-attaining students are more familiar. However, no simplistic relationship between the study of advanced mathematics and conditional inference behavior is reported by Inglis & Simpson (2008) comparing the inferences drawn from abstract conditional statements by advanced mathematics students and well-educated arts students.

Students have great difficulty accepting that ‘p **only if** q’ is logically equivalent to ‘if p then q’. One reason may be that in certain real-world situations the statements are not interchangeable. Many cannot make the distinction between the connective and the turnstile or the material implication and entailment. Being the least intuitive of all the connectives, the conditional is a source of great confusion.

3. Quantification.

Quantification, important in reasoning as well as in mathematics, often remains implicit which leads to several misconceptions and lack of understanding. Most relevant to calculus and relational contexts, it provokes many challenges to students. Students with strong mathematical background are more prepared to handle quantification and the other way round.

CONCLUSIONS

Our study provided some evidence that a course in logic across the high school curriculum provides an excellent opportunity to develop students’ high-level reasoning skills so that they correctly apply them to actual reasoning situations. Mathematicians (and other teachers) ought to be completely secure about their own logical machinery to effectively teach their students. The study presented preliminary findings from implementing in a classroom the proof opportunities designed in the curriculum.

True, the transfer from the logic taught, such as first-order logic with quantifiers, to logic applied in understanding the deductive structure of mathematics and constructing proofs doesn’t happen automatically. Yet many educators suggest students must be made familiar with the principles of mathematical argumentation, that is, they must be taught proof. Hanna & de Villiers (2007) suggest that we need more research to support or disconfirm the notion that teaching students the principles of formal logic increases their ability to prove or to understand proofs. Our study is an attempt toward that end. More analysis is needed to establish that the ability to think logically entails the ability to do mathematical proof.

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