DESIGNING INTERCONNECTING PROBLEMS THAT SUPPORT DEVELOPMENT OF CONCEPTS AND REASONING

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In this paper I illustrate the process of designing a problem which can be repeatedly used by teachers in different mathematical courses and at various levels of complexity. The same problem can support reasoning appropriate for the context in which the problem is presented. Gradual increase of the requirements for rigor from one level to the next supports the learner's development within her natural sequence of learning modes from experimental to theoretical. The course of formalization of reasoning also affects the conceptualization process related to the object of the problem.

INTRODUCTION

Development of reasoning skills and formation of concepts is a life-long process. In particular, many mathematical concepts emerge from a child's earlier experiences in a primitive form and develop further as the child has a chance to perceive and act on physical objects, to form mental images and models, and eventually reflect, categorize, and hypothesize further properties expressed in a symbolic form. At each level of concept development, a child exhibits reasoning behaviour with the degree of rigor appropriate to the level of concept maturity. According to Bruner (1968) the sequence of learning modes, enactive-iconic-symbolic, characterizes not only grade school students but also an older learner. This idea is also consistent with van Hiele's theory of learning geometry by advancing through the stages from visualization to analysis, to informal or formal deduction and finally to rigor.

In mathematical instruction, one way to reflect this long-term continuous development of mathematical thinking is to consider the notion of an *interconnecting problem*. An interconnecting problem is characterized by the following properties (Kondratieva 2011): (1) It allows simple formulation; (2) It allows various solutions at both elementary and advanced levels; (3) It may be solved by various mathematical tools from distinct mathematical branches, which leads to finding multiple solutions, (4) It is used in different grades and courses and can be discussed in various contexts.

It is proposed (Kondratieva 2011) that a long-term study of a progression of mathematical ideas revolved around one interconnecting problem is useful for developing a perception of mathematics as a connected subject by all learners. Due to the wide range of difficulty levels of its solutions, the same interconnecting problem may appear at the elementary school level, and then in progressive grades at the secondary or even tertiary level. The students, familiar with the problem from their prior hands-on experience, will use their intuition to support more elaborate techniques presented symbolically in the upper grades.

This article aims to discuss the process of creating an interconnecting problem with particular attention to the development of reasoning skills and the notion of proof. Here a Problem Designer is a mathematics teacher or mathematics curriculum developer who proceeds via the following stages: (A) choosing an initial question; (B) tailoring questions to elementary approaches; (C) upgrading to more advanced techniques; and (D) finding contexts suitable for the identified approaches and techniques within the overall curriculum.

The key point is that a problem becomes interconnecting as the Problem Designer herself experiments with the problem, identifies the types of reasoning emerging from her experiments, and starts to see different facets of its implementation in the classroom.

THERETICAL FRAMEWORK

Knowing and proving are synonyms in mathematics (Rev 1999; Balacheff 2010). On one hand, proof, as mean for validation, reinforces precise and highly logical way of thinking based on axioms, definitions, and statements, which link and describe the properties of mathematical objects. On the other hand, proofs include mathematical methods, concepts, and strategies also applicable in problem solving situations (Hanna & Barbeau 2010). Despite their central role in mathematics, it was observed that proofs receive insufficient appreciation and epistemological understanding from grade school students (and even their teachers), who often rely on empirical evidence rather that on formal deductions of mathematical theorems (Coe & Ruthven 1994). This situation identifies the needs for “problems and mathematical activities that could facilitate the learning of mathematical proof” and “designing the situations so that … the theoretical posture demonstrates all its advantages.” “The challenge is to better understand the didactical characteristics of the situation and propose a reliable model for their design”(Balacheff 2010, p. 133). One possible approach “is centred around the idea that inventing hypotheses and testing their consequences is more productive … than forming elaborate chain of deductions”(Jahnke, 2007 p.79). The process of making conjectures and inventing hypotheses requires mathematical intuition, which develops through students’ experiences not only in formal logical manipulations but also in experimental explorations of objects and ideas (De Villiers 1999). Thus collecting empirical evidence (e.g. constructing and measuring) is an important part of the mathematical education of students, and it should not be rejected as such. Instead, a productive way of incorporating experimentation and proving needs to be found so that “*proofs do not replace measurements but make them more intelligent*” (Janhke 2007, p.83). The students should gradually move from everyday thinking in terms of “open general statements” (whose domains of validity are not completely specified) towards mathematical thinking where precision is achieved at the price of cutting ties to empirical reality. This move is possible due to several roles (besides validation statements) that proofs may play in mathematical thinking (Hanna 2000; De Villiers 1999). First, at the informal deduction stage, proof as explanation of empirical observations is most appropriate. Next, students “should build a small network of theorems based on empirical evidence” and become accustomed to “*hypothetico-deductive method* which is fundamental for scientific thinking” (Jahnke 2007, p.83). At this stage, the proof functions as a “*systematization* (the organization of various results into a deductive system of axioms, major concepts and theorems)” and “*construction* of an empirical theory”. These two stages prepare students to move towards rigorous proofs aiming at establishing truth by deduction or “*incorporation* of well-known facts into a new framework” (Hanna 2000, p. 8).

The development of reasoning skills by the proposed scenario has an essential contribution in concept formation. As discussed in an upcoming publication (Tall et al), a child’s conceptual system evolves from the stage where several properties of an object occur simultaneously to the stage where these properties are linked by cause-effect relationship. This process results in developing *crystalline concepts* (e.g. platonic objects) with equivalent properties linked by mathematical proofs. Thus the process of maturation of reasoning skills both leads to and requires the use of more formal and structured conceptualizations of empirical objects.

From the perspective of this paper, two further ideas are of particular importance. First, learning to prove is a gradual process which requires years of mutually enhancing empirical and theoretical practices leading to concept formation as more properties, representations and relationships are being understood over time. If this structure is imposed on a learner in its final form, the effect of concept formation by the learner may not be achieved (Freudenthal 1971). Second, teachers’ epistemological beliefs and their abilities to model the process of proving are decisive for students’ growth in this respect. Thus, teachers’ professional preparation, which facilitates them in transitioning from empirical arguments to proof, is essential (Stylianides & Stylianides 2009). With this in mind, we now examine an example of designing an interconnecting problem.

AN EXAMPLE OF INTERCONNECTING PROBLEM DESIGN

In Euclidean geometry an isosceles triangle is often defined as a triangle which has two equal sides. It is well known that there are many equivalent characterizations of an isosceles triangle, such as “two angles are equal”, “an angular bisector is also a median”, “two altitudes are equal”, “two bisectors are equal”, each of which reflects the axial symmetry of the triangle. Proofs that the properties are pair-wise equivalent constitute problems of various levels of difficulty and contribute to building the conceptual understanding of the object by the learner.

Some properties of an isosceles triangle do not characterize it, however. It is an important exercise to recognize when this happens. For instance, think about the following **Problem:**

Observation: Consider any isosceles triangle ABC, where AB = AC. Let D be a point on BC such that AD is the angular bisector of BAC. Let M and N be midpoints of sides AB and AC respectively. Then DM = DN.

Question: In a triangle ABC with angular bisector AD and midpoints M and N of sides AB and AB, let the segments DM and DN have equal length. Does this property imply that ABC is isosceles?

In further subsections I discuss how to make this problem interconnecting in view of properties (1)-(4) and stages (A)-(D) of the design process outlined in the introduction.

A. The initial choice of problem

The choice of the problem may be justified by several factors such as how fundamental are the objects involved in the problem, the importance of the problem in the development of strands prescribed by the curriculum, or motivational aspects (e.g. surprising result).

For instance, our Problem was chosen by the Problem Designer because it deals with isosceles triangle, the object that appears in many problem-solving situations in geometry. The Problem poses a concrete question which prepares the learner to distinguish between equivalent statements and implications, and further between necessary and sufficient conditions in more abstract theorems. This problem calls for a proof involving construction of a counter-example. When such an example is constructed, it may surprise the students and produce a cognitive shift towards understanding the concept of isosceles triangle in a wider space of its examples and non-examples.

A problem must allow a simple formulation in order to become an interconnecting one. The students should understand the question and be able to specialize and exemplify the statement of the question (Mason et al 1982). For example, in our Problem we start with an Observation which can be justified by the symmetry argument or even by simply folding the paper triangle along its axis of symmetry. In order to answer the Question students may try other examples of triangles. They will quickly realize that they have to either find an example of non-isosceles triangle with given property or proof that such a triangle does not exist. It is clear what one has to do, but not obvious how one can approach this problem. A systematic search for an example needs to be initiated by the solver.

B. Making the problem Interconnecting: elementary level

First, the Problem Designer puts herself in the position of problem solver. She starts from thinking how she can approach the problem at the most elementary level. According to Bruner’s classification, this corresponds to enactive representation of the problem and involves the use of real objects and manipulatives. For a modern learner equipped with a computer, thinking at the elementary level also involves the access to virtual manupulatives, and experimentations within dynamic geometry software (DGS) environments. Through the use of dragging function, the learner receives a visualization of her problem as continuum of options. The example she is looking for may be just one static picture in this continuum.

Construction of such a continuum requires understanding of many basic mathematical ingredients of the problem as well as the properties of the software tools. Here is a protocol from a problem designer's attempt to solve the problem.

> First, I examined initial configuration. I have an isosceles triangle ABC, where AB = AC. Here M is the midpoint of AB, N is the midpoint of AC. Line AD is the angular bisector of angle BAC. We know that in this case DM = DN due to symmetry argument. I draw this triangle on the screen (Fig. 1). Points M and N lie on the circle with centre at D. But this circle also intersects the extension of side AC at point L, which means that DM=DL. Aha, I have an idea: point L could be the midpoint of the side of the required non-isosceles triangle. Now I place point F on the extension of AC such that AL = LF and look at the triangle ABF. Denote by E the intersection point of BF and the extension of angular bisector AD. If I could drag points and change the figure in such a way that D coincides with E then I will complete the task.

The Problem Designer experiments with this figure but unfortunately it does not seem to be possible to complete the task within this particular construction and she proclaims:

> Maybe there is no such example at all. But then I have to explain why. Perhaps I should try to construct something else. Maybe I should not start with an isosceles triangle at all.

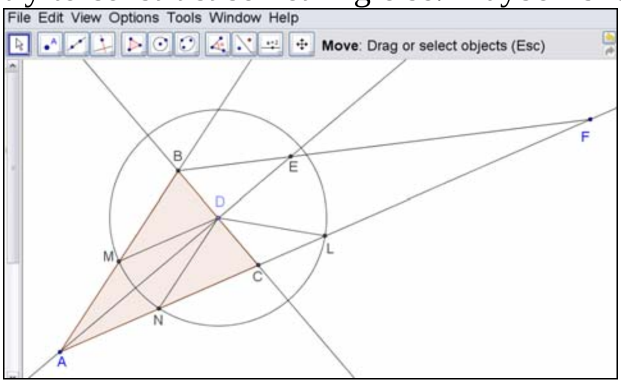

Figure 1: Unsuccessful attempt to build an example.

Meantime she also learned that the software has an option "reflect a point with respect to another point" which she uses to place a vertex, knowing the position of a midpoint. She continues to build her example. Finally, she succeeds in doing so, still employing the idea that the second point of intersection of the circle with the angle side is the key of the construction.

> I start with an arbitrary angle with vertex at A. I place two arbitrary points M and K one on of each of the sides (see Fig 2, left). Now I place point B on one side such that AM = MB and place point C on the other side such that AK = KC. The intersection of BC with

the angular bisector is called D. I draw the circle with centre at D and radius DM. This circle intersects side AC at points N and L. Now I want to make K coinciding with either N or L. I conjecture that the former case gives me an isosceles triangle and the latter, if this is possible, will produce the required example.

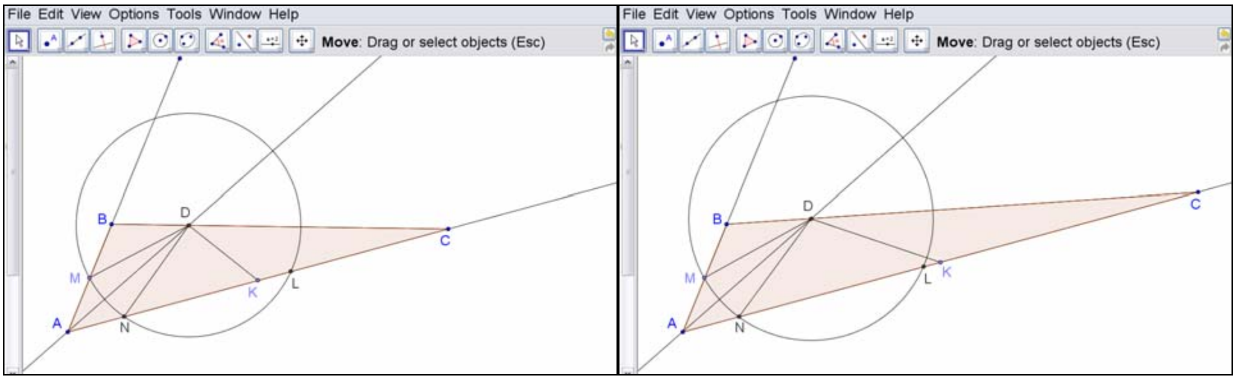

Figure 2: A successful attempt to build an example: dragging K along the side AC.

By dragging point K along the side AC I can interchange the positions of point K and L on the side of the triangle (see Figure 2, right). Thus, by dragging K along the side I achieve that points K and L coincide. (See Figure 3, left). And now I confirm that if K coincides with N then we indeed obtain an isosceles triangle (Figure 3, right).

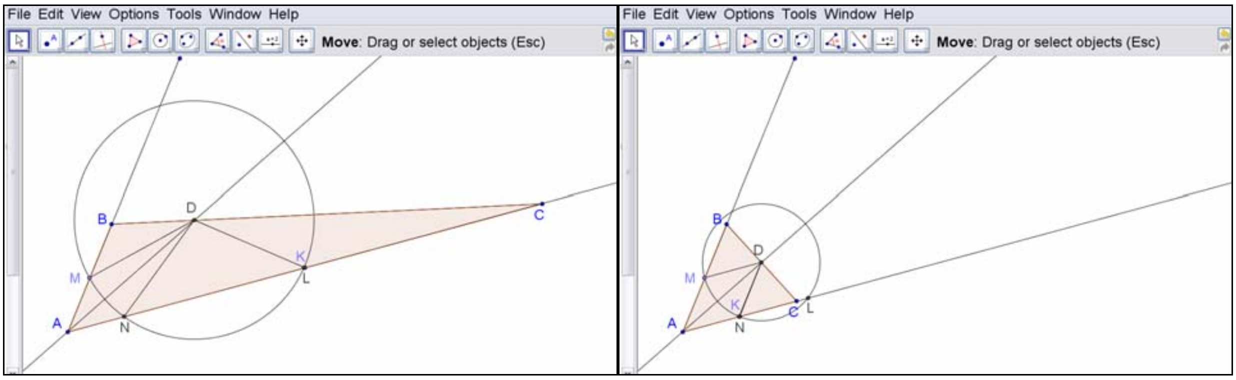

Figure 3: A successful attempt to build an example of a triangle where DM = DN = DK.

Once the problem is understood at the elementary level, the Problem Designer thinks how this can be used for introduction of more advanced techniques.

C. Connecting a problem to more advanced mathematics

Now, the Problem Designer aims to use intuition developed through the visualization of a solution for constructing a symbolic solution, which would correspond to the highest stage in Bruner's classification. In secondary school students learn equations of lines and circles. They also learn the idea that solving a system of two equations representing these curves gives the coordinates of the points of intersection of the curves. Thus to make the problem interconnecting one may try to represent previously obtained geometric solution algebraically. Here is what the Problem Designer does next.

> I was looking at the Figures 2 and 3 and understood that to make an algebraic representation I need to introduce Cartesian coordinates. Let the side AC lie along the *x*-axis with A at the origin. Then point K has coordinates (k,0). I introduce a general equation of the sides and the bisector; I fix coordinates of M and B and find how change of k affects the coordinates of D, C and L. I will set coordinates of L and K equal and find k from this equation. This is a plan. … But its implementation becomes very cumbersome! I do not think that students will benefit from it.

Then she introduces another approach aimed at easier “algebraization”.

> In a coordinate system I draw two rays starting from the origin and symmetric with respect to the x-axis (see Fig 4, left). I pick an arbitrary point M on the upper ray and point G on the x-axis. The circle with centre at G and radius GM intersects the lower ray at two points. I devote by N the one which is not symmetric to M with respect to x-axis. Now I place point B on the upper ray such that AM=MB, and point C on the lower ray such that AN=NC. The segment BC intersects x-axis at point D. I can drag point G along the x-axis in the position of point D and thus I obtain the example (see Fig. 4, right).

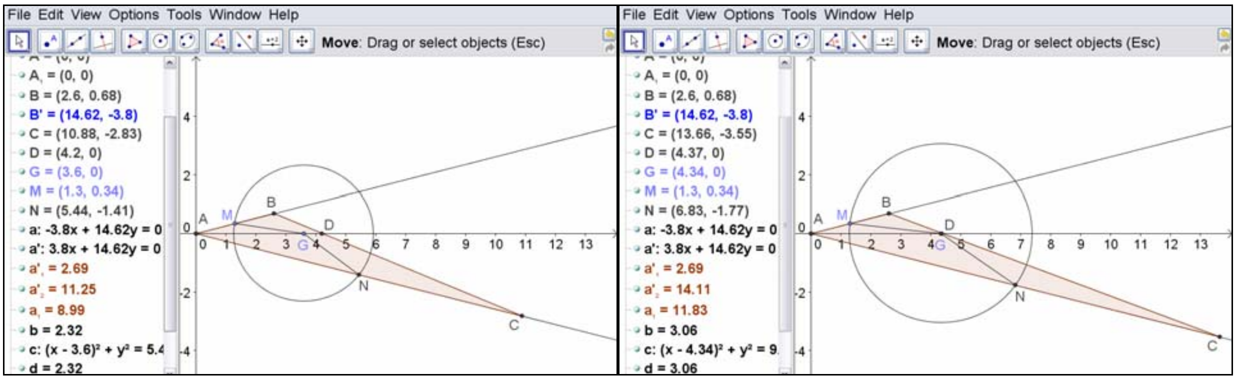

Figure 4: A successful attempt to build an example, which allows “algebraization”.

Now I construct an algebraic model. The two rays have linear equations in the form $y = mx$ and $y = -mx$, where the slope m can vary. Point G has coordinates $(g,0)$ and the circle has equation $(x-g)^2 + y^2 = r^2$, where $r =$ GM. In order to find points of intersections of the ray and the circle we need to solve the quadratic equation $(x-g)^2 + (mx)^2 = r^2$, that is to express roots x_1, x_2 via g and r. Once I find the roots, I obtain coordinates of all points in terms of them: M(x_1, mx_1), B$(2x_1, 2mx_1)$, N$(x_2, -mx_2)$, C$(2x_2, -2mx_2)$. Now, I want point G belong to the segment BC. Equating slopes of BC and BG gives the condition: $g = \dfrac{4x_1x_2}{x_1 + x_2}$. But from the quadratic equation I find that $x_1x_2 = \dfrac{g^2 - r^2}{m^2 + 1}$ and $x_1 + x_2 = \dfrac{2g}{m^2 + 1}$, thus I obtain the relation between the radius and the coordinate of the centre of the circle $g=\sqrt{2}r$. From this relation I find integer coordinates for the vertices of a non-isosceles triangle for which the property DM = DN can be verified. For instance, if A(0,0), M(3,1), B(6.2), N(15,-5), C(30,-10), D(10,0) then DM=DN=$\sqrt{50}$.

D. Identifying levels and context suitable for problem

Once several approaches to solve our Problem are identified, the Problem Designer needs to summarize and identify all possible places in the curriculum where this problem potentially belongs. For example, the teacher may give students complete freedom in the choice of approaches to the question. Then the teacher may introduce a DGS and let students to try to build their examples from scratch. During this activity students rethink their task in view of tools available in the computerised environment. For example, the process of constructing angular bisector or a midpoint of a segment may be an automated part of a DGS, and then the major construction pertinent to the problem is conceptualized in terms of these operations.

Alternatively, the teacher may give the students an applet such as shown in Figures 2-3, which forces the students to explain already prefabricated construction. The students may be asked to state their observations about what constrains are preserved in the applet and how important they are for building the example, or what objects are introduced and what role in the solution they play. The students shall articulate their conjectures about observed relationships, for instance "What kinds of examples are possible (e.g. acute, obtuse, right angle)?" They also may be asked to explain why the example they construct with the applet is not just a visual illusion or an approximation; how do they know that the real example with all required characteristics exists. (The role of misleading diagrams in Geometry in relation to proofs is discussed in Kondratieva, 2009).

The problem may appear in the view of the students again when they study the coordinate approach. This time an applet from Figure 4 will be useful because the drawing explicitly reveals the coordinate system and equations on the side suggest a more general algebraic approach. Here the students may be asked to use concrete equations of the lines and circle and then generalize them and analyse the situation in an algebraic form, returning to concrete examples provided by the applet for a verification of their general course of reasoning. Consideration of various cases may be supported by the applet as well.

The problem may be recalled once again when the analysis of quadratic equations is discussed, and the existence of two real roots may be related to the existence of two points of intersection of the circle with the side of the angle. Since traditionally the number of real roots is said to be defined by the discriminant of a quadratic equation, the students may be asked to investigate the connection between the parameters of the figure (radius of the circle in Fig 4) and the resulting coefficients affecting the sign of the discriminant, and draw their conclusions with justification.

Finally, the problem can serve as an illustration in the study of conditional statements (implications) and their converses, inverses and contrapositives in formal logic, as well as a study of proofs by counter-example. Now the emphasis can be made on the logical structure of the statements because the details of building the example are familiar to the students from previous encounters with the problem.

CONCLUSION.

In this paper, I illustrate the process of designing an interconnecting problem by using an example from Euclidean geometry. While making our Problem interconnecting, a variety of instances supporting the development of reasoning and proving skills has occurred. Being placed in the domain of mathematical activity the Problem deals with concrete objects, their properties and relationships. The experimentations with dynamic geometry software especially when the students interact with figures that are constrained to retain certain properties, forces the students to explain their actions and observations, to make and justify their conjectures. This activity accompanied by the requirement to systematise observed results pushes the learner towards the hypothetico-deductive stage of reasoning.

An algebraic approach is introduced after geometrical meaning of the model has been understood. While at this stage the focus is on setting and solving equations and development of algebraic thinking, the experience within DGS environments supports students' reasoning as they visualize the situation hidden behind variables and equations. These visualizations contribute into development of learners' intuition as well as in forming algebraic-geometric connections and perhaps a more holistic view on mathematics itself.

Behind concrete problems in mathematics often there is a more general and far-reaching agenda. For example, our Problem aims at grasping the general notion of a universally valid statement by making sense of the proclamation "For every isosceles triangle the segments DN and DM are equal." This statement has the same form as "For every isosceles triangle with AB=AC the angles B and C are equal." However, the Question is posed to identify whether this property of an isosceles triangle necessarily defines an isosceles triangle. It invites students to realise that not every property of an object in fact defines the object. This fact should be brought to the students' attention forcing them to distinguish between equivalent conditions, such as "equal sides" and "equal angles", and those for which implication works only one way and not both ways. By solving this problem students not only advance their concept of an isosceles triangle, but also build their understanding of the statement's generality, the nature of implication, the notions of necessary versus sufficient conditions, and the idea of proof by counter-example. Thus, the design of an interconnecting problem discussed above aims at fostering reasoning skills at both visual-empirical and symbolic-theoretical levels within the same mathematical question in the background. Further research on teaching practices that involve adoption or design of interconnecting problems and their affect on students' reasoning abilities will show to what extent this would indeed be possible to achieve.

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