PROCESSES OF PROOF WITH THE USE OF TECHNOLOGY: DISCOVERY, GENERALIZATION AND VALIDATION IN A COMPUTER MICROWORLD

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Our general concern is to investigate the role that new technologies play in the development of processes of proof. We use Balacheff's work (1987, 1999) to talk of pragmatic proofs vs. intellectual and formal proofs. We present part of a case-study from a larger research where students used a Logo-based microworld for the exploration of infinite sequences and series, to illustrate how some of the elements that computer microworld explorations, activities and visual means bring, can lead to a process of discovery and acceptance of mathematical results, and create stepping-stones (pragmatic proofs) in the development of mathematical proofs.

INTRODUCTION AND THEORETICAL FRAMEWORK

We are interested in investigating the role that new technologies play in the stages of proving in mathematics education. Research in this direction is often related to geometry and the influence of dynamic geometry systems on the learning of mathematical proof (Hoyles & Jones, 1998; Villiers, 1998, Balacheff, 1999). However, in this paper we draw data from a different kind of study where students used a Logo-based microworld for the exploration of infinite sequences and series. We aim to illustrate how the environment and its tools gave students enough means of mathematical exploration and expression through which they could discover patterns, make generalizations and conjectures, and validate their results.

We begin with a brief discussion on the function of proof as a means of understanding, and the role of technology for developing a sense of proof and intuition before a formal proof is presented. We then present a brief summary of some important concepts developed by Balacheff (1987, 1999) that will serve as part of our theoretical framework. Finally, we present sample data from our study to illustrate the ways in which computational experiences can function as a means of creating understanding and acceptance of mathematical results, thus constituting processes of proof.

The construction of meanings through proofs, and the role of technology

The function of proof as "a key tool for the promotion of understanding" has been stressed by Hanna (1995). While Hanna defends the value of formal proof, she also makes a distinction between the function of proof in mathematics and that in mathematics education:

> while in mathematical practice the main function of proof is justification and verification, its main function in mathematics education is surely that of explanation … It might be a calculation, a visual demonstration, a guided discussion observing proper rules of

argumentation, a preformal proof, an informal proof, or even a proof that conforms to strict norms of rigour … Clearly the challenge is to have [students] understand *why* [results] are true. (p. 47)

Other studies have highlighted features that are making clear the complex field of mathematical proofs: the role played by empirical evidence in contrast with deductive arguments (Chazan, 1993); the difference between argumentative reasoning and deductive reasoning (Duval, 1991); and students' proof schemata (Harel, 1996).

With the advent of new technologies, the empirical methods of mathematics research have been revitalized. Mathematicians themselves are recognizing that technology is changing the way we approach proving. Thurston (1994), who analyses the nature of proof and of mathematics itself, also emphasizes that it is a search for *understanding* which is at the basis of the exploration and logical processes leading to a proof; and he advocates the use of computers for exploration and discovery of mathematical ideas, giving priority to what he calls "humanly understandable" proofs over formal proofs.

Other authors and researchers have investigated the role of computer-based explorations and visualization to develop a sense of proof and give intuition for formal proof (for instance, by building up an overall picture of the relationships involved), considering these important complementary elements to mathematical proofs (e.g. Barwise & Etchemendy, 1991). Among those are Cuoco & Goldenberg (1992) who consider that the activity of constructing proofs involves a research technique where conjectures arise through the combination of experimentation and deduction.

One of the difficulties with proving in school mathematics is that concepts which expert mathematicians regard as intuitive are not intuitive to students since intuition depends on previous experience (Tall, 1991). To overcome this, Tall (op.cit., p.118) considers that:

> by introducing suitably complicated visualizations of mathematical ideas it is possible to give a much broader picture of the ways in which concepts may be realized, thus giving much more powerful intuitions than in a traditional approach. … intuition and rigour need not be at odds with each other. By providing a suitably powerful context, intuition naturally leads into the rigour of mathematical proof.

The components of proving

Balacheff (1987, 1999) has introduced certain ideas that we consider important for our analysis. He distinguishes between *pragmatic proofs* and *intellectual proofs*; emphasizing the role of language in the passage from the former to the latter. Pragmatic proofs are those based on effective action carried out on the representations of mathematical objects. They lead to practical knowledge that the subject can use to establish the validity of a proposition. Intellectual proofs demand that such knowledge is reflected upon, and their production necessarily requires the use of language that expresses (detached from the actions) the objects, their properties and their relationships. In other words, pragmatic proofs are based on action, while the use of a functional language (which includes a specific vocabulary and symbolism) and a "mental experience" (where actions are interiorised) characterize the transition to the intellectual ones. The transition from pragmatic proofs to intellectual proofs culminating in mathematical proof, involves three components: the *knowledge* or *levels-of-action component* (the nature of knowledge: knowledge in terms of practices — "*savoir-faire";* knowledge as object; and theoretical knowledge); the *language* or *formulation component* (ostentation, familiar language, functional language, formal language); and the *validation component* (the types of rationale underlying the produced proofs: from pragmatic, to intellectual, to mathematical proofs).

COMPONENTS OF PROOF IN A LOGO MICROWORLD

Here, we present an example to illustrate how certain computational activities constitute pragmatic proofs, with progress on the level of the first component (knowledge), and some progress in the other components. Although we cannot provide evidence on how computational activities can lead to the development of more rigorous intellectual proofs, we believe that progress achieved in earlier stages constitute suitably powerful experiences that create familiarity and understanding of the problem (an intuition) necessary for transition to the latter (cf. Tall, 1991; see above). It is likely one of the kind of experiences incorporating exploratory computer and visual activities considered useful and advocated by some of the researchers reviewed above (e.g. Tall, 1991; Thurston, 1994; Cuoco & Goldenberg, 1992), and which provides elements of proof as "explanatory" (Hanna, 1995).

This example (from a case-study of a pair of high-school students) is taken from a larger research that investigated the mediating role of a Logo microworld designed for the exploration and study of infinite sequences and series, in students' conceptions of infinity and infinite processes. That study involved detailed case studies of 4 pairs of students working and interacting with the microworld. (Students worked in pairs to facilitate the sharing and discussion of ideas —simultaneously providing the researcher with insights into their thinking processes— and give them independence from the instructor). Each pair of students, previously instructed in Logo programming, worked with one computer for five 3-hour sessions. To facilitate the analysis of students' experiences, we worked with only one pair of students at a time, using a clinical interview style. The role of the researcher was that of a participant observer, suggesting the field of work (the initial procedures and activities), as well as new ideas for exploration when needed, yet allowing students to be in control of the explorations, giving them freedom to explore and express their ideas. Students were informally interviewed throughout the sessions but formal interviews were conducted at the beginning and end of the study.

As part of the microworld activities, students explored sequences such as $\{1/2^n\}$, $\{1/3^n\}$, $\{(2/3)^n\}$, $\{2^n\}$, and $\{1/n\}$, $\{1/n^{1.1}\}$ and $\{1/n^2\}$, and the sequences of their corresponding partial sums. They wrote Logo procedures to construct visual models of those sequences, using Logo's turtle geometry: spirals (Fig.1) (where each successive "piece" of the spiral represents a term of the sequence, so that the total length of the spiral corresponds to that of the sum of the terms, i.e. the corresponding series), bar graphs (Fig.2), staircases, and straight lines (where again, the total length represents the corresponding series or partial sum of the terms of the sequence).

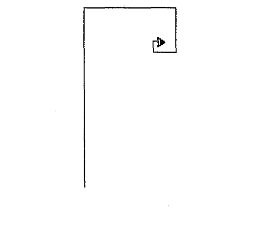

Fig.1: Spiral model for the sequence $\{1/2^n\}$

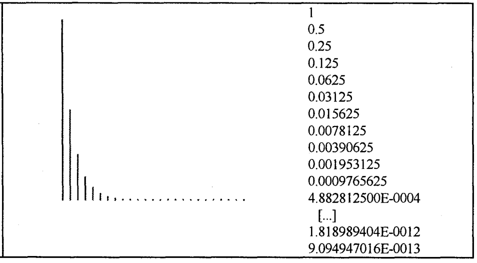

Fig.2: Bar graph model for the sequence $\{1/2^n\}$ with numeric output

For example, one of the initial Logo procedures used in these activities was of the form:

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TO DRAWING :L
IF :L < 1 [STOP]
MODEL
DRAWING (FUNCTION :L )
END
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Where MODEL described the steps for building a model such as a spiral (FD :L RT 90) or a bar graph (FD :L JUMP), and FUNCTION described the operation on :L at each step (e.g. :L/2).

Thus, the different visual models for the same sequence provided different perspectives of the same process. As the explorations progressed students added Logo commands to their procedures that would allow them to carry out a complementary analysis of the numerical values (to further analyse their behaviour, and the apparent limits, if any existed or appeared to exist). Through the observation of the visual (and numeric) behaviour of the models, students were able to explore the convergence, and the type of convergence, or divergence, of a sequence and its corresponding series.

The microworld was designed to simultaneously provide its users with insights into a range of infinity-related ideas, and offer the researcher a window (cf Noss & Hoyles, 1996) into the users' thinking processes. The microworld provided a means for students to actively construct and explore different types of representations (symbolic, visual and numeric) of infinite sequences via programming activities. In general, the computer setting provided an opportunity to analyse and discuss in conceptual (and concrete) terms the meaning of a mathematical situation. For example, drawing a geometric figure using the computer necessitated an analysis of the geometric structure under study and an analysis of the relationship between the visual and symbolic representations.

What is relevant for us here, is that the exploratory setting of the microworld activities allowed students to engage in an active process of discovery of the properties and characteristics of the processes under study. That is, the environment seems to have provided a language for asking questions, as well as tools for exploring these questions. In many cases students found what seemed like patterns and properties, which led them to formulate and test conjectures, as well as articulate relationships and build generalizations.

There were several levels in which the microworld activities functioned as stepping-stones towards a proof. At a first level, students made observations and discoveries situated within the medium of the microworld. We have described elsewhere (e.g. Sacristán, 1997) how students were able to discover patterns related to the processes under study but situated within the context of the microworld environment. By playing in, interacting with, and working within the microworld, they could express themselves through the tools, activities and forms of symbolism built into the environment. At a second level, some students were able to abstract and articulate their findings in a way that could be taken beyond the medium in which the findings were constructed; consciously exploiting the tools of the microworld for discovery, exploration, and "pragmatic proof" (Balacheff, op.cit.) of mathematical relationships or "theorems".

Pragmatic proofs: discovery, generalization and validation in the microworld

Through the sample data below, taken from one of the case-studies (a pair of 16-17 year-old boys: Manuel and Jesús), we aim to illustrate, in particular, how students made predictions and then tested their validity by using all the available tools in the microworld.

During the second worksession of explorations, as described above, of sequences of the type $\{(1/k)^n\}$, these students discovered that the series of the type $\sum_{n=1}^{\infty}\frac{1}{k^n}$, where the integer[1] $k>1$, converge to $\frac{1}{k-1}$. These students had been exploring and comparing the sequences $\{1/2^n\}$, $\{1/3^n\}$ and began to discover a pattern in the behaviour of the corresponding series: Manuel observed that as they increased the denominator value k in the sequences of the type $\left\{\frac{1}{k^n}\right\}_n$, then the limit of the corresponding series was smaller and in fact seemed to have as value $\frac{1}{k-1}$. They explicitly constructed a generalization for this mathematical result (which later they would call proudly "their theorem") and used it to *predict* the probable behaviour of other sequences and series of the same type:

Manuel: Look, if you subtract 1 from the number that is the base in the denominator, and you divide 1 by that number, then that is the number to which it will approach. If we do it with 3: 3 minus 1 is 2, and it tends to a half... So if it was $1/2000^N$, the sum must approach 1/1999...

Jesús: Yes, the bigger the base in the denominator, the smaller the limit.

[1] Manuel and Jesús seemed to consider k implicitly as a positive integer larger than 1, although they did not make this condition explicit.

Manuel: But now we have a method for knowing to where it approaches.... We saw that $1/2^N$ became small very quickly, but the case of $1/3^N$ decreases much much more quickly. And we saw that its series didn't tend to 1 like the previous one, that it approached a half, so we noticed a more or less regular behaviour, so if we wanted to know to how much the series of $1/2000^N$ would be we would only have to reduce it by a number, and it would tend to 1/1999.

Manuel and Jesús then employed the medium and its tools to test out their predictions, looking for a proof of their conjecture. They began by changing the sequence generating function to $1/4^N$, predicting that the corresponding series would tend to 1/3. They used all the resources available to explore this sequence and its series, looking at all the available visual models (the Spiral, Stairs, Bar Graph and Line models). They were amazed at how quickly the values of the sequence decreased. From the rapidly decreasing behaviour of this sequence they deduced that the corresponding series converged, which they visually verified, using the Line model. Although the visual explorations were enough to convince the students of the validity of their conjecture, they complemented these with a numeric exploration of the partial sums (they built a procedure which computed the value of a partial sum). They observed that the 20th partial sum printed out to be 0.3333333333, confirming further their hypothesis. A final test of their conjecture was carried out by exploring the sequence $\{1/13^n\}$, through visual and numeric representations — which showed the much more rapid decrease of this sequence — and again verified that the corresponding series tended to the predicted value of 1/12.

For Manuel and Jesús there was now no doubt that the conjecture was true, but Manuel did worry that this mathematical generalization would not hold if the value of *k* was infinite (what Balacheff, 1987, 1999, has described as a *crucial experience*), and he looked for favourable arguments:

Manuel: Listen... there might be a contradiction in our assumption: if we did one over infinity... Ah! but infinity minus any number is still infinity..., so we are right! It tends to zero. If one over infinity tends to zero, then also one over infinity-minus-one, because infinity minus one is infinity, then it also tends to zero. And here we have that the bigger the base of the denominator gets, the smaller the series.

It is worth noting that most students, including a pair of younger students (two 14 year-old girls, one of them called Consuelo) discovered the rule for the behaviour of the series of the type $\sum \frac{1}{k^n}$, which they tested and then generalized. Manuel and Jesús were more experienced mathematically which was reflected in the way they expressed the rule, but the younger students also constructed the generalization within the context of the activity — what Noss & Hoyles (1996) have called a *situated abstraction* — expressing it relative to the inputs used by the procedure (e.g. the scale):

Consuelo: So the sum of the bars for 1/3 it's one half [the scale], and for 1/4 it would be 1/3 [of the scale], and for a fifth: 1/4 [of the scale], and so on.

DISCUSSION AND CONCLUDING REMARKS

The possibility of working with many cases (different sequences of the same type), and use diverse resources (different visual models and complementary types of representations: visual and numeric), provided students with a means to (i) infer their own generalization through the discovery of a pattern, and (ii) to validate and confirm their predictions and generalization (becoming convinced of the general validity of their conjecture). In this sense, the microworld became a mathematical laboratory. The results were not formally proven, and the students were aware of this, but the process of repeatedly observing different variations, cases, and situations, was enough to convince the students of the validity (or in other cases falseness) of their conjectures[2], constituting pragmatic proofs. Elsewhere, we called these experiences "situated proofs" (Moreno & Sacristán, 1995; Sacristán, 1997). These are pragmatic proofs that result from the combined use of all the elements available in the microworld in an attempt to confirm conjectures. Situated proofs are experiences that lead students to discover and make sense of a mathematical relationship, convincing them of its validity. In the same way as situated abstractions, these experiences are dependent on the tools of the medium (hence the term "situated"). They are therefore not yet detached enough from the representations of the objects and the actions to constitute an intellectual proof.

Nevertheless, progress is shown for all three components. The fact that students spontaneously tested their conjectures (adjusting them as a result of the explorations), in order to convince themselves (and others) of their validity, shows progress on the level of the *validation component.* On the level of the *knowledge component,* the formulation of predictions or conjectures involved a process of reflection and analysis on the part of the students, as they had to, for instance, evaluate the role and relationship of the variables involved. Also, the Logo-based activities allow students to construct certain forms of symbolism[3] to express and interact within the medium (situated abstractions); this is progress on the level of the *language component.* Thus, these experiences were often powerful enough to act, at least, as explanatory proofs (Hanna, 1995). They are pragmatic proofs that can be thought of as the collection of activities that build meaning for a theorem before a formal proof is presented.

Thus, the use of technological tools can provide a field of exploration and mathematical experimentation in which it is possible to deal with mathematical content through the representations provided by the medium. Proofs on the pragmatic level can become more powerful and enrich students' mathematical experiences by allowing them to discover results and formulate probable conjectures. In this sense, we consider that this type of

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[2] Exploration through visual and numeric representations, observation of the behaviour of the process under study, and the structure of the code, are all elements that students used to convince themselves of the convergence or divergence of a process, and/or of the existence of a limit.

[3] This contrasts with findings in another of our studies using Cabri-Géomètre, where geometrical objects are manipulated, and there is a development of geometric visualisation, but the use of the symbolic geometric language is very limited.

microworld activities can strengthen the foundations for the passage to intellectual proofs. However, more research is needed to find activities that follow or are based upon computational activities, which will lead to the transition towards intellectual and formal proofs; in agreement with Hanna (1995), the role of the teacher is likely to be crucial.

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