REASONING AND PROOF IN GEOMETRY: PREREQUISITES OF KNOWLEDGE ACQUISITION IN SECONDARY SCHOOL STUDENTS[1]

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Proof is regarded to be an important aspect in the mathematics classroom. Nonetheless, it is difficult for students to give mathematical proofs. Our research aims at identifying important dimensions of proving abilities. In a study with 669 grade seven students, we investigated how mathematical knowledge, the ability to evaluate correct and incorrect proofs, and scientific reasoning influence the students' performance in proving. Our results give evidence that these aspects contribute significantly to students' abilities.

1. Theoretical Framework

Proof and logical argumentation are important topics in mathematics as a science, and mathematics may even be regarded a proving science. The role of proof in the school curriculum did not always reflect that importance. In the 1970s and 1980s, there was an intensive discussion whether proofs should be part of the mathematics curriculum. Mathematics educators criticized that proving in the classroom emphasized formal aspects but disregarded mathematical understanding (Hanna, 1983). This is still a consensus among many mathematics educators, but its consequences have been revised. Proof is considered as an important topic of the mathematics curriculum (NCTM, 2000) and as an essential aspect of mathematical competence. Nonetheless, proof is not used as a synonym for formal proof. In particular, researchers such as Hanna and Jahnke (1993), Hersh (1993), Moore (1994), Hoyles, (1997), Harel and Sowder (1998) have pointed out that proving spans a broad range of formal and informal arguments. Understanding and generating proofs is an important component of mathematical competence, and mathematical argumentation has been identified as an essential element of higher order mathematical competence in the TIMS study (cf. Baumert, Lehmann et al., 1997). The current discussion emphasizes the development of proof concepts (Boero, 1999), the continuum from exploration to proof (NCTM, 2000) and the role of reasoning and argumentation in finding a proof (Reiss, Klieme & Heinze, 2001). On the background of the PISA findings (Deutsches PISA-Konsortium [German PISA Consortium], 2002) the individual prerequisites for secondary school students' performance in proof tasks are of particular interest.

The *Standards and Principles for School Mathematics* (NCTM, 2000) call for a "focus on learning to reason and construct proofs as part of understanding mathematics so that all students

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- recognize reasoning and proof as essential and powerful parts of mathematics;
- make and investigate mathematical conjectures;
- develop and evaluate mathematical arguments and proofs;
- select and use types of reasoning and methods of proof as appropriate."

The approach is applied to all stages of education from preschool to grade 12. In the early years informal inductive elements are underlined whereas the formal deductive elements become more important for older students.

This standard is the basis for our own work (Reiss & Thomas, 2000; Reiss, Klieme & Heinze, 2001; Reiss & Renkl, in press). With respect to the aspects described in this standard, we assessed students' performance on mathematical proofs. We concentrated on

- students' abilities for proving (i.e. the knowledge of mathematical propositions and concepts, its application to simple situations, understanding a proof, the ability to argue mathematically),
- their knowledge of proof methods and their evaluation of the correctness of proofs (cf. Healy & Hoyles, 1998),
- their abilities in the domain of scientific reasoning.

2. Design of the Study

The aspects described above were investigated in a study with students at the end of grade 7 and in the beginning of grade 8. In grade 7, students get an intensive instruction on geometry proofs and argumentation, which is continued in grade 8. The study was performed between January 2001 and February 2002. We will report on data of a pre-test, which was given to them in June 2001. The results presented here are based on the data of 669 students (363 female students, 306 male students).

We developed a test for the assessment of their abilities to prove which consisted of six items on basic qualifications and seven items on justification and reasoning. For assessing their knowledge of proof methods and their evaluation of the correctness of proofs we adapted a test constructed by Healy and Hoyles (1998). The students had to evaluate four solutions for a proof task: two incorrect ones (empirical, circular) and two correct ones (narrative, formal). In order to assess abilities in the domain of scientific reasoning we presented tasks which consisted of two parts, namely a reduction of a given problem space (Klahr & Dunbar, 1988) and ordering of information given. The test items were presented to the students as paper and pencil questionnaires in a classroom situation.

3. Results

The students performed quite well in terms of knowledge of mathematical propositions and concepts, and its application to simple situations. We observed more difficulties concerning their understanding of proofs and their abilities to reason mathematically. The variation within these components was considerable. Whereas most of the students had a basic knowledge of mathematical propositions and concepts (M=7.4, s=2.5, maximum number of points: 12), they scored lower with respect to items asking for mathematical argumentation (M=5.2, s=3.7, maximum number of points: 14).

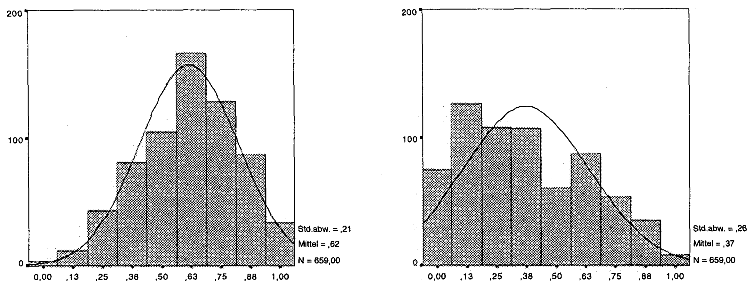

Figure 1: Distribution of norm scores for basic mathematical knowledge (left) and mathematical argumentation (right)

When constructing the test, we applied Klieme's (2000) model of mathematical competencies. The data show a close connection between this model and the actual students' performance. The following table provides information on the students' performance (table 1). According to the achievement we grouped the students into a lower third, a middle third, and an upper third and compared their performance with respect to the levels of competency.

In level of competency I, which was represented by five items, there is no formal mathematical reasoning required, just the application of concepts and rules as well as elementary inferences. In level II (three items) the students must have a sound knowledge of geometrical concepts and facts. They must be able to give a correct justification to given geometrical problems and to find an appropriate notation. In level III (four items) the students must be able to order their arguments in a meaningful way. A prerequisite for this is autonomous and creative problem solving and reasoning. Table 1 shows that the lower third of the students did not have any correct solutions in level III tasks whereas the upper third managed to solve 85% of the level I problems and 89% of the level II problems. Nearly all the other numbers are gradual increases or decreases between these extremes. This can be regarded as an internal validation of the test.

	Level of competency I Simple application of rules and elementary reasoning M=0.69	Level of competency II Argumentation and justification (one step) M=0.56	Level of competency III Argumentation and justification (several steps) M=0.24
Lower third of the students (N=238)	51%	22%	0%
Middle third of the students (N=225)	72%	61%	18%
Upper third of the students (N=206)	85%	89%	50%

Table 1: Percentage of correct solutions

It should be mentioned that we found large differences between the 27 classes involved in the study. Their average scores ranged from very low (M=5.7) to appropriate (M=17.7) given a maximum of 26 points for the 13 items. In the three classes showing the lowest average scores the students achieved less than half of the possible points (with the exception of only one person). Accordingly, these classes lack of high achieving students, whereas in the classes with the highest average scores there was a wide variation of achievement levels. The high achieving classes solved the simple tasks in a good or even very good manner, the low achieving classes performed satisfactorily on these items. However, we found differences at tasks, in which argumentation skills were needed for the solution. The students of the low achieving classes hardly had any correct solutions with respect to these items. The responses show that the students had the declarative knowledge to solve these items, but they were not able to apply it correctly. These differences might be caused by teaching styles (cf. Baumert, Lehmann et al., 1997). We will investigate these differences in an ongoing video study.

Concerning the students' knowledge of proof methods and their evaluation of the correctness of proofs, it can be seen that students have difficulties in identifying incorrect solutions as incorrect. It is significantly easier for them to classify correct solutions as correct ($p<.001$). Moreover, for the students it is easier to evaluate proofs than to formulate proofs by themselves. Concerning norm scores we found an average score of M=0.67 (s=0.33) for the knowledge of proof methods and the evaluation of the correctness of proofs. The norm score for the test items of level II and level III is M=0.37 (s=0.26). This difference is highly significant. Comparing the lower third, medium third, and upper third with respect to the achievement in the test on mathematical performance, there are highly significant differences concerning the students' knowledge of proof methods and their evaluation of the correctness of proofs.

In the domain of scientific reasoning we found that students are often guided by plausible arguments even if these arguments are not logically consistent. A high proportion of the students could not solve tasks in which plausibility was not a hint for the solution (cf. table 2). If students could not solve a plausible task, they could not solve tasks presented in an unusual context either. On the other hand, if they could solve plausible tasks they had a high probability to solve tasks presented in an unusual context.

	Tasks with a unusual context not solved	Tasks with a unusual context partly solved	Tasks with a unusual context solved
Plausible tasks not solved	163	22	11
Plausible tasks partly solved	111	54	43
Plausible tasks solved	80	56	119

Table 2: Distribution of solutions in items with unusual and with plausible contexts

We expected a correlation between abilities in the domain of scientific reasoning and the achievements in the area of justification and proving (cf. table 3). This expectation was confirmed.

	Level of competency I Simple application of rules and elementary reasoning M=0.68	Level of competency II Argumentation and justification (one step) M=0.56	Level of competency III Argumentation and justification (several steps) M=0.24
No formal strategy available (N=354)	67%	54%	20%
Rudimentary formal strategy available (N=132)	66%	55%	21%
Complete formal strategy available (N=173)	74%	60%	33%

Table 3: Percentage of each level of competence dependent on the availability of formal solution strategies in tasks concerning scientific reasoning

The data show that students who are more successful using formal strategies (namely students who do not need a plausible context but are able to solve problems in an unusual context) have higher scores with respect to items of competency level II and competency level III.

Discussion

There are a number of abilities, which may enable students to perform proofs. In our study there is evidence that the following abilities play an important role in this context:

Basic knowledge of mathematical facts and argumentation:

The students need an appropriate level of knowledge about simple mathematical propositions and concepts. Most students have those basic abilities, although the level was quite different in the various classes investigated. Moreover, students need higher order skills in order to apply their knowledge in a proof context. This is more difficult for the students, thus only a few of them are able to apply their knowledge. These findings are in accordance with results of Klieme, Neubrand and Lütge (2001) within the PISA study. Moreover, our data could be validated by Klieme's (2000) model of mathematical competency.

Knowledge of proof methods and evaluation of the correctness of proofs:

It was more difficult for the students to identify incorrect solutions as faulty than to identify correct solutions as being correct. This confirms findings of a former study with grade 13 students (Reiss, Klieme & Heinze, 2001) and findings of Healy and Hoyles (1998) as well as Küchemann and Hoyles (2001). Moreover, we found a correlation between this knowledge and the achievement measures on basic mathematical knowledge of facts and argumentation.

Scientific reasoning:

Plausibility may be regarded as guiding principle for scientific reasoning. Many students rely on plausible argumentation when confronted with reasoning tasks. The plausibility of propositions is more important for their argumentation than facts, which determine that a line of thought is logical or not logical. If students show a high level of scientific reasoning they usually show good achievement in proof-related tasks. Accordingly, scientific reasoning seems to be closely related to mathematical abilities.

These three aspects will only partly explain the interindividual variance in argumentation, reasoning, and proof. We could identify significant differences between mathematics classrooms. Accordingly, the differences in the students' performance cannot be explained by individual prerequisites alone. This is in accordance with Bronfenbrenner's theory of ecological development. Bronfenbrenner (1979) suggests, to take into account the context of teaching and learning, which includes the teacher and the social context. Our findings suggest that a significant portion of achievement differences in the mathematics classroom will be explained by these variables.

Our future research (and the second part of the study described here) will concentrate on the influence of teachers and school variables in mathematics education with respect to argumentation and proof. Instruments will be questionnaires concerning the teaching styles, which are presented to students and teachers, questionnaires concerning the mathematical beliefs of students and teachers, and video observations of mathematics classrooms.

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