



The International Commission on Mathematical Instruction (ICMI)

ICMI Study 19: Proof and Proving in Mathematics Education

Discussion Document

- I. Rationale for the study
- II. Themes of the study (including specific questions)
- III. Design of the study
- IV. Participation in the study

I. RATIONALE

Mathematics educators face a significant task in getting students to understand the roles of reasoning and proving in mathematics. This challenge has now gained even greater importance as proof has been assigned a more prominent place in the mathematics curriculum at all levels. The recent National Council of Teachers of Mathematics (NCTM) Principles and Standards document and several other mathematics curricular documents have elevated the status of proof in school mathematics in several educational jurisdictions around the world.

This renewed curricular emphasis on proof has provoked an upsurge in research papers on the teaching and learning of proof at all grade levels. This re-examination of the role of proof in the curriculum and of its relation to other forms of explanation, illustration and justification (including dynamic graphic software) has already produced several theoretical frameworks, giving rise to many discussions and even heated debates. An ICMI Study on this topic would thus be both useful and timely.

An ICMI Study on proof and proving in mathematics education would necessarily discuss the different meanings of the term proof and bring together a variety of viewpoints. Proof has played a major role in the development of mathematics, from the Euclidean geometry of the Greeks, through various forms of proofs in different cultures, to twentieth-century formal mathematics based on set-theory and logical deduction. In professional mathematics today, proof has a range of subtly different meanings: for example, giving an axiomatic formal presentation; using physical conceptions, as in a proof that there are only five Platonic solids; deducing conclusions from a model by using symbolic calculations; or using computers in experimental mathematics. For mathematicians, proof varies according to the discipline involved, although one essential principle underlies all its varieties:

To specify clearly the assumptions made and to provide an appropriate argument supported by valid reasoning so as to draw necessary conclusions.

This major principle at the heart of proof extends to a wide range of situations outside mathematics and provides a foundation for human reasoning. Its simplicity, however, is disguised in the subtlety of the deep and complex phrases “to specify the assumptions clearly”, “an appropriate argument” and “valid reasoning”.

The study will consider the role of proof and proving in mathematics education, in part as a precursor for disciplinary proof (in its various forms) as used by mathematicians but mainly in terms of developmental proof, which grows in sophistication as the learner matures towards coherent conceptions. Sometimes the development involves building on the learners’ perceptions and actions in order to increase their sophistication. Sometimes it builds on the learners’ use of arithmetic or algebraic symbols to calculate and manipulate symbolism in order to deduce consequences. To formulate and communicate these ideas require a simultaneous development of sophistication in action, perception and language.

The study’s conception of “developmental proof” has three major features:

1. Proof and proving in school curricula have the potential to provide a long-term link with the discipline of proof shared by mathematicians.
2. Proof and proving can provide a way of thinking that deepens mathematical understanding and the broader nature of human reasoning.
3. Proof and proving are at once foundational and complex, and should be gradually developed starting in the early grades.

A major classroom role for proof is essential to maintaining the connection between school mathematics and mathematics as a discipline. Although proof has not enjoyed the same degree of prominence in mathematical practice in all periods and contexts, and although standards of rigour have changed over time, proof undoubtedly lies at the heart of mathematics.

Similarly proof and proving are most properly used in the classroom to promote understanding, which in no way contradicts their role in mathematics. Mathematical proof consists, of course, of explicit chains of inference following agreed rules of deduction, and is often characterised by the use of formal notation, syntax and rules of manipulation. Yet clearly, for mathematicians proof is much more than a sequence of correct steps; it is also and, perhaps most importantly, a sequence of ideas and insights with the goal of mathematical understanding -- specifically, understanding why a claim is true. Thus, the challenge for educators is to foster the use of mathematical proof as a method to certify not only that something is true but also why it is true.

Finally, the learning of proof and proving in school mathematics should be developmental and should start in the early grades. The success of this process would clearly depend on teachers’ views about the essence and forms of proofs, on what

teachers do with their students in classrooms, on how teachers interpret and implement curricular tasks that have the potential to offer students opportunities to engage in proving, and on how they diagnose students' difficulties in proving and design instructional interventions to help overcome these difficulties.

II. THEMES OF THE STUDY

The ICMI Study will be organised around themes that provide a broad range of points of view on the teaching and learning of proof in various contexts, whether symbolic, verbal, visual, technological or social. Within each of the themes, the following issues are of utmost importance:

1. Teachers' views and beliefs
2. Teachers' preparation and professional development
3. Curriculum materials and their role in supporting instruction

Below, we describe some of the themes and suggest a number of related research questions. Contributions on each theme should address these specific questions but need not be limited to them, so long as any additional questions raised are relevant to that theme.

1. Cognitive aspects

Cognitive aspects of proof cover the entire development of proof and proving, from the young child to the research mathematician. They range from the manner in which the growing person develops a proving attitude to convince the self and others, through the initial use of specific examples, through prototypical numerical and visual examples representing broader classes of instances, to formal axiomatic proofs widely acceptable to the mathematical community. While proofs are considered either valid or invalid, the development of proof, both in the growing child and in the research of mathematicians, involves arguments that carry various levels of conviction that are not absolute. For example, Tall's framework of worlds —of conceptual embodiment, proceptual symbolism and axiomatic formalism—suggests a dynamic development of proof through embodiment and symbolism to formalism. For instance, the formula for the sum of the first n whole numbers can be proved from a specific or generic picture, from a specific, generic or algebraic sum, from a practical potentially infinite form of induction, from a finite axiomatic form of induction from the Peano postulates, or even from a highly plausible visual demonstration. This part of the study will consider various theories of cognitive aspects of proof.

Possible questions about cognitive aspects:

1. Is it possible/preferable to classify forms of proof in terms of *cognitive development*, rather than just in terms of *type of proof* (e.g., by exhaustion, contradiction, induction)?

2. When we classify proof cognitively, can we look from the learners' viewpoint as they grow from the elementary grades to university, rather than just from the expert's viewpoint, and appropriately value their current ways of proving?
3. How do we encompass empirical classifications of proof processes within a coherent cognitive development (which may differ for different individuals)?
4. How can teachers and mathematics educators use our knowledge about learners' cognitive development to develop ways of teaching proof that take account of each learner's growing ways of proving?
5. What are learners' and teachers' beliefs about proof, and how do they affect the teaching and learning of proof?
6. What theoretical frameworks and methodologies are helpful in understanding the development of proof from primary to tertiary education, and how are these frameworks useful in teaching?

2. Argumentation and proof

Understanding the relationship between argumentation (a reasoned discourse that is not necessarily deductive but uses arguments of plausibility) and mathematical proof (a chain of well-organised deductive inferences that uses arguments of necessity) may be essential for designing learning tasks and curricula that aim at teaching proof and proving. Some researchers see mathematical proof as distinct from argumentation, whereas others see argumentation and proof as parts of a continuum rather than as a dichotomy. Their different viewpoints have important didactical implications. The first group would focus mainly on the logical organisation of statements in a proof and would aim to teach a conceptual framework that builds proof independent of problem solving. On the other hand, the second group would focus primarily on the production of arguments in the context of problem solving, experimentation and exploration, but would expect these arguments to later be organized logically so as to form a valid mathematical proof.

From a very young age, students show high degrees of ability in reasoning and in justifying their arguments in social situations; however, they do not naturally grasp the concept of mathematical proof and deductive reasoning. Therefore, educators must help students to reason deductively and to recognize the value of the concept of mathematical proof. Some educators hold the traditional assumption that teaching students elements of formal logic, such as first-order logic with quantifiers, would easily translate into helping them to understand the deductive structure of mathematics and to write proofs. However, research has shown that this transfer doesn't happen automatically. It remains unclear what benefit comes from teaching formal logic to students or to prospective teachers, particularly because mathematicians have readily admitted that they seldom use formal logic in their research. Hence, we need more research to support or disconfirm the notion that teaching students formal logic increases their ability to prove or to understand proofs.

Possible questions about argumentation and proof:

1. How can we describe the argumentative discourses developed in mathematics teaching? What is the role of argumentation and proof in the conceptualization process in mathematics and in mathematics education?
2. Within the context of argumentation and proof, how should mathematics education treat the distinction that logicians and philosophers make between truth and validity?
3. To what extent could focussing on the mathematical concept of implication in both argumentation and proof contribute to students' better grasp of various kinds of reasoning?
4. How can educators make explicit the different kinds of reasoning used in mathematical proof and in argumentative discourse (e.g., *Modus Ponens*, exhaustion, disjunction of cases, *Modus Tollens*, indirect reasoning etc.)?
5. Quantification, important in reasoning as well as in mathematics, often remains implicit. To what extent does this lead to misconceptions and to lack of understanding?
6. How can teachers deal with the back-and-forth between conjectures and objects or between properties and relations involved in the exploration of mathematical objects? To what extent does this exploration help students understand the necessity of mathematical proof rather than just argumentation?
7. Are we justified in concluding that logic is useless in teaching and learning proof just because many mathematicians claim that they do not use logic in their research? What kind of research program could be developed to answer this question?
8. What are the relationships between studies on argumentation and proof by researchers from other disciplines, e.g., logicians, philosophers, epistemologists, linguists, psychologists and historians, and research in mathematics education?
9. What conditions and constraints affect the development of appropriate situations for the construction of argumentation and proof in the mathematics classroom?
10. Which learning environments and activities help to improve students' ability in argumentation and proof?

3. Types of proof

Some aspects of the study might deal with types of proof characterized by their mathematical or logical properties, such as specific proof techniques, (e.g., proof by exhaustion, proof by mathematical induction, proof by contradiction) or proofs of specific types of claims (e.g., existence proofs, both constructive and non-constructive).

These different types of proof (or techniques of proving) may have many diverse pedagogical properties and didactic functions in mathematics education. A case in point is inductive proof (proof by example), which is frequently the only type of proof comprehensible to beginners; it may be mathematically valid (e.g., for establishing existence or for refutation by counterexample) or invalid (e.g., supportive examples for a

universal statement). Another type, generic (or transparent) proof, is infrequently used but may have high didactic potential.

The various ways of proving, such as verbal, visual or formal, may be a factor in understanding proofs and in learning about proving in general. Specific proofs may lend themselves particularly well to specific ways of proving.

Possible questions about types of proof:

1. To what extent, and at which levels of schooling, is it appropriate to introduce specific proof techniques? What are the particular cognitive difficulties associated with each type of proof?
2. Is it important to introduce proof in a diversity of mathematical domains and which proofs are more appropriate in which domains?
3. At which level and in which curricula is it relevant to introduce the notion of refutation? In particular, when should one raise the question of what is needed to prove or refute an existential claim as opposed to a universal one?
4. How and at which stage should teachers facilitate the transition from inductive proof (proof by example) to more elaborate forms of proof?
5. What status should be given to generic proof? How can the properties of generic proofs be used to support students' transition from inductive to deductive proof?
6. At which level, and in which situations, should the issue of the mathematical validity or lack of validity of inductive proofs be discussed, and how?
7. To what extent and how is the presentation of a proof (verbal, visual, formal etc.) relevant in understanding it and in learning about the notion of proof generally?
8. To what extent is the presentation of a proof (in)dependent of the nature of the proof? Do some proofs lend themselves particularly well to specific presentations? For example, can visual theorems have non-visual proofs?
9. Do students perceive different types of proofs as more or less explanatory or convincing?

4. Dynamic Geometry Software and Transition to Proof

Both philosophers and psychologists have investigated the connection between deductive reasoning and argumentation. However, there is still no consensus on the exact nature of this connection. Meanwhile some researchers have looked for possible mediators between plausible argumentation and mathematical proof. The main didactical problem is that at first glance there seems to be no natural mediator between argumentation and proof. Hence, the problem of continuity or of discontinuity between argumentation and proof is relevant for research and for teaching of proof.

Dynamic Geometry Software (DGS) fundamentally changes the idea of what a geometric object is. DGS can serve as a context for making conjectures about geometric objects and thus lead to proof-generating situations. Specifically, it can play the role of mediator in the transition between argumentation and proof through its 'dragging function', thanks to its instant feedback and to the figures created on the screen as a result of the dragging

movements. The dragging function opens up new routes to theoretical knowledge within a concrete environment that is meaningful to students. For example, it can introduce seemingly infinite examples to support a conjecture or it can help in showing students degenerate examples or singular counterexamples to a statement (e.g., when a given construction that works for building a figure degenerates into singular cases, producing a different figure). Moreover, while dragging, pupils often switch back and forth from figures to concepts and from abductive to deductive modalities, which helps them progress from the empirical to the theoretical level. The different modalities of dragging can be seen as a perceptual counterpart to logical and algebraic relationships. In fact, dragging makes the relationships between geometric objects accessible at several levels: perceptual, logical and algebraic.

Possible questions about DGS environments:

1. To what extent can explorations within DGS foster a transition to the formal aspects of proof? What kinds of didactical engineering can trigger and enhance such support? What specific actions by students could support this transition?
2. How could the issues of continuity/discontinuity among the different phases and aspects of the proving processes (exploring, conjecturing, arguing, proving etc.) be addressed in DGS environments?
3. To what extent can activities within DGS environments inhibit or even counter the transition to formal aspects of proof?
4. What are the major differences between proving within DGS environments and proving with paper and pencil?
5. How can the teacher handle the different modalities of proving (induction, abduction, deduction etc.) that explorations in DGS environments may generate?
6. How can DGS help in dealing with proofs by contradiction or proofs by example, given that through dragging one could get ‘infinite examples’, degenerate examples or the singular counterexamples to a statement?
7. How can DGS environments be used for approaching proofs not only in geometry but also in other subjects, such as algebra and elementary calculus?
8. What are the significant differences among different DGSs used in teaching proof?
9. What are the main differences between DGS environments and other technological environments (software other than DGS, concrete materials, mathematical machines, symbolic computation systems etc.) in tackling the issue of proof in the classroom? Can a multiple approach, which suitably integrates different environments, be useful for approaching proof?

5. The Role of Proof and Experimentation

The traditional view of proof has ignored the role of experimentation in mathematics and has perceived the verification of mathematical statements as the only function of proof. However, in recent years several authors have emphasized the intimate relationship between proof and experimentation, as well as the many other important functions of proof within mathematics besides verification: explanation, discovery, intellectual

challenge, systematization etc. Moreover, research in dynamic geometry has shown that, despite obtaining a very high level of conviction by dragging, students in some contexts still display a strong cognitive need for an explanation of a result; that is, why it is true. Such a need gives a good reason for the introduction of proof as a means of explaining why a result is true.

However, not all new results in mathematics are discovered through experimentation. Deductive reasoning from certain givens can often directly lead to new conclusions and to new discoveries through generalization or specialization. In this context, proof takes on a systematizing role, linking definitions, axioms and theorems in a deductive chain. Likewise, experimentation in mathematics includes some important functions relevant to proof: conjecturing, verification, refutation, understanding, graphing to expose mathematical facts, gaining insights etc. For example, mathematicians can formulate and evaluate concept definitions on the basis of experimentation and/or formal proof, as well as comparing and selecting suitable definitions on the basis of criteria such as economy, elegance, convenience, clarity etc. Suitable definitions and axioms are necessary for deductive proof in order to avoid circular arguments and infinite regression. Thus, the establishment of a mathematical theorem often involves some dynamic interplay between experimentation and proof.

The relationship between proof and experimentation poses a general didactical and educational research question: How can we design learning activities in which students can experience and develop appreciation for these multi-faceted, inter-related roles of proof and experimentation? This in turn comprises several additional questions.

Possible questions about proof and experimentation:

1. How can teachers effectively use the explanatory function of proof to make proof a meaningful activity, particularly in situations where students have no need for further conviction?
2. How can students' abilities to make their own conjectures, critically evaluate their validity through proof and experimentation, and produce counter-examples if necessary be stimulated and developed over time?
3. How can teachers and mathematics educators develop effective strategies to help students see and appreciate the discovery function of proof -- for example, deriving results deductively rather than experimentally or from deriving further unanticipated results and subsequent reflections on those proofs?
4. What are students' natural cognitive needs for conviction and verification in different mathematical contexts, with different results and at different levels? How can these needs be utilized, changed and developed through directed instructional activities so that students appreciate the verification function of proof in different contexts?
5. What arguments can teachers use in school and university to foster students' appreciation of the meaning of proof and to motivate students to prove theorems?
6. What type of 'guidance' is needed to help students eventually produce their own independent proofs in different contexts?

7. Rather than just providing them with pre-fabricated mathematics, how do we involve students in the deductive systematization of some parts of mathematics, both in defining specific concepts and in axiomatizing a piece of mathematics? How able are students to identify circular arguments or invalid assumptions in proofs and how do we develop these critical skills?

6. Proof and the Empirical Sciences

Frequently, students do not see a connection between argumentation in empirical situations and mathematical proof. They consider proof a mathematical ritual that does not have any relevance to giving reasons and arguments in other circumstances or disciplines. However, mathematical proof is not only important in mathematics itself but also plays a considerable role in the empirical sciences that make use of mathematics.

Empirical scientists put up hypotheses about certain phenomena, say falling bodies, draw consequences from these hypotheses via mathematical proof and investigate whether the hypotheses fit the data. If they do, we accept the hypotheses; otherwise we reject them. Thus, in the establishment of a new empirical theory the flow of truth provided by a mathematical proof goes from the consequences to the assumptions; the function of a proof is to test the hypotheses. Only at a later stage, after a theory has been accepted, does the flow of truth go from the assumptions to the consequences as it usually does in mathematics. These considerations suggest a series of questions for investigation.

Possible questions about proof and the empirical sciences:

1. To what extent should mathematical proofs in the empirical sciences, such as physics, figure as a theme in mathematics teaching so as to provide students with an adequate and authentic picture of the role of mathematics in the world?
2. Would insights about the role of proof in the empirical sciences be helpful in the teaching of geometry, given that geometry deals with empirical statements about the surrounding space as well as with a theoretical system about space?
3. Could insights about the complex role of proof in the empirical sciences be helpful in bridging students' perceptual gap between proof and proving in mathematics and argumentation in everyday life?
4. To what extent and how should philosophers of mathematics, mathematics educators and teachers develop a unified picture of proving and modelling, which are usually considered completely separate topics in mathematics?
5. Could a stronger emphasis on the process of establishing hypotheses (in the empirical sciences) help students better understand the structure of a proof that proceeds from assumptions to consequences and thus the meaning of axiomatics in general?
6. To what extent does a broader conception of proof require the collaboration of mathematics and science teachers?

7. Proof at the Tertiary Level

At the tertiary level, proofs involve considerable creativity and insight as well as both understanding and using formal definitions and previously established theorems. Proofs tend to be longer, more complex and more rigorous than those at earlier educational levels. To understand and construct such proofs involves a major transition for students but one that is sometimes supported by relatively little explicit instruction. Teachers increasingly use students' original proof constructions as a means of assessing their understanding. However, many questions remain about how students at the tertiary level come to understand and construct proofs. Here we lay some of the questions out clearly, proposing to examine them in the light of both successful teaching practices and current research.

Possible questions about proof at the tertiary level:

1. How are instructors' expectations about students' performance in proof-based mathematics courses different from those in courses students experienced previously?
2. Is learning to prove partly or even mainly a matter of enculturation into the practices of mathematicians?
3. How do the students conceive theorems, proofs, axioms, definitions and the relationships among them? What are the students' views of proof and how are their views influenced by their experiences with proving?
4. What are the roles of problem solving, heuristics, intuition, visualization, procedural and conceptual knowledge, logic and validation?
5. What previous experiences have students had with proof that teachers can take into consideration?
6. How can we design opportunities for student teachers to acquire the knowledge (skills, understandings and dispositions) necessary to provide effective instruction about proof and proving?

III. DESIGN OF THE STUDY

The ICMI Study on the role of proof and proving in mathematics education will consist of three components: 1) an invited *Study Conference*, 2) a *Study Volume* and 3) a *Study Website*.

1) The *Study Conference* will be held in Taipei, Taiwan, from May 10 to May 15, 2009.

As is the normal practice for ICMI studies, participation in the study conference is by invitation to the authors of accepted contributions. The number of participants will be limited to approximately 120. We hope that the conference will attract not only “experts” but also some “newcomers” to the field with interesting, refreshing ideas or promising work in progress, as well as participants from countries usually under-represented in mathematics education research meetings. Unfortunately, an invitation to participate in the conference does not imply financial support from the organisers; participants should

finance their own attendance. We are seeking funds to provide partial support for participants from non-affluent countries, but the number of such grants will be limited.

The *Study Conference* will be a working one; every participant will be expected to be active. We therefore hope that the participants will represent a diversity of backgrounds, expertise, experience and nationalities.

The printed proceedings, available at the conference, will contain the refereed submissions of all participants and will form the basis of the study's scientific work.

2) The *Study volume*, a post-conference publication, will appear in the New ICMI Study Series (NISS). Participation in the conference does not automatically ensure inclusion in the book. The Study volume will be based on selected contributions as well as on the outcomes of the Conference. The exact format of the Study volume has not yet been decided. We expect it to be an edited book which can serve as a standard reference in the field.

3) The *Study website*, <http://jps.library.utoronto.ca/ocs/index.php?cf=8> accessible before, during and after the conference, will contain information on the conference and will be updated periodically.

A report on the *Study* and its outcomes will be presented at the 12th International Congress on Mathematical Education in 2012.

IV. PARTICIPATION IN THE STUDY

Call for contributions

The International Program Committee (IPC) invites individuals or groups to submit original contributions. A submission should represent a significant contribution to knowledge about learning and teaching proof. It may address questions from one or more of the study themes, or further issues relating to these, but it should identify its primary focus. New researchers in the field and participants from countries under-represented in mathematics education research meetings are especially encouraged to submit contributions. We hope that researchers and mathematics educators from the early years to tertiary levels, as well as mathematicians, will come up with new insights and guidelines for future work.

The format of papers must be as follows:

1. A maximum of 6 pages, including references and figures.
2. Written in English using Times New Roman 14-point font, 16-point line space, and 6 points between paragraphs; occupying a frame of 170 by 247 mm.
3. The title (in 16-point bold capitals), author(s) name(s) (in 14-point bold), and affiliation(s) of author(s) (in 14-point italics) should appear in this order centered, all in Times New Roman.

4. The paper must begin with an abstract of up to 10 lines, single-spaced, in italics.
5. Video clips may be referred to in the paper and a link should be provided.
6. A template will be posted on the conference website.

Further technical details about the format of submissions will be available on the Study website <http://jps.library.utoronto.ca/ocs/index.php?cf=8>, which will be progressively updated with all study and travel information.

Review process

Submitted contributions will be reviewed and selected on the following criteria: (a) clear links to the *Study's* goals; (b) explicit fit with one or more of the themes; (c) clear structure and exposition; (d) potential to contribute to the quality and advancement of the Study.

Study timeline

- Submissions for participation in the Study should be uploaded to the website by June 30, 2008.
- Submissions will be reviewed and decisions made about inclusion in the conference proceedings. Notifications about these decisions will be sent by November 15, 2008 to all those who sent in submissions.

International Program Committee

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