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The Teaching of Proof

Authors: Deborah Loewenberg Ball (section 2)[1], Celia Hoyles (section 3)[2], Hans Niels Jahnke (chair, section 5)[3], Nitsa Movshovitz-Hadar (section 4)[4]

Abstract

This panel draws on research of the teaching of mathematical proof, conducted in five countries at different levels of schooling. With a shared view of proof as essential to the teaching and learning of mathematics, the authors present results of studies that explore the challenges for teachers in helping students learn to reason in disciplined ways about mathematical claims.

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97C30, 97C50, 97D20. Proof, didactics of mathematics, mathematical reasoning

1. Introduction

Proof is central to mathematics and as such should be a key component of mathematics education. This emphasis can be justified not only because proof is at the heart of mathematical practice, but also because it is an essential tool for promoting mathematical understanding.

This perspective is not always unanimously accepted by either mathematicians or educators. There have been challenges to the status of proof in mathematics itself, including predictions of the 'death of proof.' Moreover, there has been a trend in many countries away from using proof in the classroom (for a survey see Hanna & Jahnke, 1996).

In contrast to this, the authors of the present paper agree that proof must be central to mathematics teaching at all grades. Nevertheless, there are lessons to be learned from the debates over the role of proof. For many pupils, proof is just a ritual without meaning. This view is reinforced if they are required to write proofs according to a certain pattern or solely with symbols. Much mathematics teaching in the early grades focuses on arithmetic concepts, calculations, and algorithms, and, then, as they enter secondary school, pupils are suddenly required to understand and write proofs, mostly in geometry. Substantial empirical evidence shows that this curricular pattern is true in many countries.

Needed is a culture of argumentation in the mathematics classroom from the primary grades up all the way through college. However, we need to know more about the difficulties pupils encounter when they are confronted with proof and the

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[1] School of Education, University of Michigan, Ann Arbor, MI 48109-1259, USA, dball@umich.edu

[2] Institute of Education, University of London, 20 Bedford Way, London WC1H OAL, UK, choyles@ioe.ac.uk

[3] Fachbereich Mathematik und Informatik, Universität Essen, 45117 Essen, Germany, njahnke@uni-essen.de

[4] Department of Education in Science and Technology, Technion - Israel Institute of Technology, Haifa 32000, Israel, nitsa@tx.technion.ac.il

challenges faced by teachers who seek to make argumentation central to the mathematics classroom. The epistemological difficulties that confront students in their first steps into proof can be compared to those faced by scientists in the course of developing a new theory. At the beginning, definitions do not exist. It is not clear what has to be proved and what can be presupposed. These problems are interdependent, and researchers (like students) find themselves in danger of circular reasoning. In the infancy of a theory, a proof may serve more to test the credibility or the fruitfulness of an assumption than to establish the truth of a statement. Only later, when the theory has become mature (or the student has come to feel at home in a domain), can a proof play its mathematical function of transferring truth from assumptions to a theorem.

All in all, work is needed in three areas with regard to the teaching of proof. We need (1) a more refined perception of the role and function of proof in mathematics, including studies of the practices of proving in which active mathematicians engage (epistemological analysis), (2) a deeper understanding of the gradual processes and complexities involved in learning to prove (empirical research) and (3) the development, implementation and evaluation of effective teaching strategies along with carefully designed learning environments that can foster the development of the ability to prove in a variety of levels as from the primary through secondary grades and up to college level (design research).

We begin in Section 2 with an analysis of what mathematical proof might involve in the primary grades. Section 3 gives results of a longitudinal study on the development of proving abilities in grades 8 and 9. Section 4 is based upon an empirical investigation of college level teaching and shows how the natural habit of referring to an example can be used as a leverage into the teaching of proof, and section 5 discusses the idea of 'physical mathematics' as an environment for the teaching of proof.

2. What does it take to (teach to) reason in the primary grades?

Although the teaching and learning of mathematical reasoning has often been seen as a focus only beginning in secondary school, calls for improvements in mathematics education in the U.S. have increasingly emphasized the importance of proof and reasoning from the earliest grades (NCTM, 2000, p. 56). While some may regard such a focus on reasoning and proof secondary to the main curricular goals in mathematics at this level, we consider reasoning to be a basic mathematical skill. Yet what might 'mathematical reasoning' look like with young children, and what might it take for teachers to systematically develop students' capacity for such reasoning? These questions form one strand of our research on the teaching and learning of elementary school mathematics.

We define 'mathematical reasoning' as a set of practices and norms that are collective, not merely individual or idiosyncratic, and that are rooted in the discipline (Ball & Bass, 2000, 2002; Hoover, in preparation). Mathematical reasoning can serve as an instrument of inquiry for discovering and exploring new ideas, a process that we call the *reasoning of inquiry*. Mathematical reasoning also functions centrally in justifying or proving mathematical claims, a process that we call the *reasoning of justification*. It is this latter on which we focus here.

The reasoning of justification in mathematics, as we see it, rests on two foundations. One foundation is an evolving *body of public knowledge*—the mathematical ideas, procedures, methods, and terms that have already been defined and established within a given reasoning community. This knowledge provides a point of departure, and is available for public use by members of that community in constructing mathematical claims and in seeking to justify those claims to others. For professional mathematicians, the base of public knowledge might consist of an axiom system for some mathematical structure simply admitted as given, plus a body of previously developed and publicly accepted knowledge derived from those axioms. Hence, the base of public mathematical knowledge defines the grain size of the logical steps which require no further warrant, that is acceptable within a given context. The second foundation of mathematical reasoning is *mathematical language*—symbols, terms, notation, definitions, and representations—and rules of logic and syntax for their meaningful use in formulating claims and the networks of relationships used to justify them. 'Language' is used here to refer to the entire linguistic infrastructure that supports mathematical communication with its requirements for precision, clarity, and economy of expression. Language is essential for mathematical reasoning and for communicating about mathematical ideas, claims, explanations, and proofs. Some disagreements stem from divergent or unreconciled uses of terminology, whereas others are rooted in substantive and conflicting mathematical claims (Crumbaugh, 1998; Lampert, 1998). The ability to distinguish these requires sensitivity to the nature and role of language in mathematics.

We have been tracing the development of mathematical reasoning in a class of Grade 3 students (ages 8 and 9) across an entire school year using detailed and extensive records of the class: videotapes of the daily lessons, the students' notebooks and tests, interviews with students, and the teacher's plans and notes. By comparing the class's work at different points in time, we are able to discern growth in the students' skills of and dispositions toward reasoning. We offer two brief examples here. Early in the school year, the teacher presented the problem, 'I have pennies (one-cent coins), nickels, (five-cent coins), and dimes (ten-cent coins) in my pocket. Suppose I pull out two coins, what amounts of money might I have?' The children worked to find solutions to this problem: 2¢, 6¢, 10¢, 11¢, 15¢, and 20¢. The teacher asked the students whether they have found <u>all</u> the solutions to the problem, and how they know. Some students seemed uncertain about the question. Other students offered explanations: 'If you keep picking up different coins, you will keep getting the same answers,' 'If you write down the answers and think about it some more until you have them all.' The students believed they had found them all, but it was because they could not find any more. Their empirical reasoning satisfied them. Moreover, they had neither other ideas nor methods for building a logical argument which would allow them to prove that this problem (as worded) had exactly six solutions. They also did not have the mathematical disposition to ask themselves about the completeness of their results when working on a problem with finitely many solutions.

In contrast, consider an episode four months later. Based on their work with simple addition problems, the third graders had developed conjectures about even and odd numbers (e.g., an odd number plus an odd number equals an even number). They generated long lists of examples for each conjecture: 3 + 5 = 8, 9 + 7 = 16, 9 + 9 = 18, and so on. Two girls, amidst this work, argued to their classmates: 'You can't prove that Betsy's conjecture (*odd + odd = even*) always works. Because, um, . . . numbers go on and on forever, and that means odd numbers and even numbers go on forever, so you couldn't prove that *all* of them work.' The other children became agitated and one of them pointed out that no other conclusion that the class had reached had met this standard. Pointing to some posted mathematical ideas, the product of previous work, one girl questioned: 'We haven't even tried them with all the numbers there is, so why do you say that those work? We haven't tried those with all the numbers that there ever could be.' And other children reported that they had found many examples, and this showed that the conjecture was true. But some were worried: One student pointed out that there are 'some numbers you can't even pronounce and some numbers you don't even know are there.' A day later, however, challenged by the two girls' claim, the class arrived at a proof. Representing an odd number as a number than can be grouped in twos with one left over, they were able to show that when you add two odd numbers, the two ones left over would form a new group of two, forming an even sum:

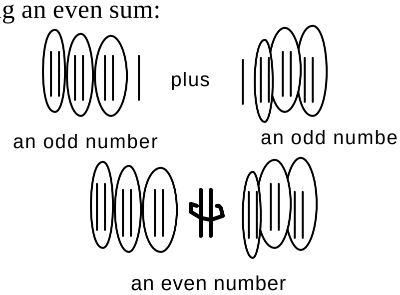

Figure 1: odd + odd = even

This episode illustrates the important role for definitions. Having a shared definition for odd and even numbers enabled these young students to establish a logical argument, based on the structure of the numbers. As one girl explained to her classmates: 'All odd numbers if you circle them by twos, there's one left over, so if you . . . plus another odd number, then the two ones left over will group together, and it will make an even number.' The definition equipped them to transcend the barrier that 'numbers go on forever' because it afforded them the capacity for quantification over an infinite set. Moreover, this episode shows the students having developed in their ability to construct, inspect, and consider arguments using previously established public mathematical knowledge.

Our research on the nature, structure, and development of mathematical reasoning has made plain that mathematical reasoning can be learned, and has highlighted the important role played by the teacher in developing this capacity. Three domains of work for the teacher have emerged from our analyses. A first concerns the selection of mathematical tasks that create the need and opportunity for substantial mathematical reasoning. The two-coin problem, for instance, did not originally require students to find all the solutions. Asking this transformed an ordinary problem into one that involved the need to reason mathematically about the solution space of the problem. The second domain of teachers' work centers on making mathematical knowledge public and in scaffolding the use of mathematical language and knowledge. Making records of the mathematical work of the class (through student notebooks, public postings, etc.) is one avenue, for it helps to make that work public and available for collective development, scrutiny, and subsequent use. This includes attention to where and in what ways knowledge is recorded, as well as how to name or refer to ideas, methods, problems, and solutions. Making mathematical knowledge and language public also requires moving individuals' ideas into the collective discourse space. A third domain of work, then, concerns the establishment of a classroom culture permeated with serious interest in and respect for others' mathematical ideas. Deliberate attention is required for students to learn to attend and respond to, as well as use, others' solutions or proposals, as a means of strengthening their own understanding and the subsequent contributions they can make to the class's work.

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3. The complexity of learning to prove deductively

Deductive mathematical proof offers human beings the purest form of distinguishing right from wrong; it seems so transparently straightforward – yet it is surprisingly difficult for students. Proof relies on a range of 'habits of mind' – looking for structures and invariants, identifying assumptions, organising logical arguments – each of which, individually, is by no means trivial. Additionally these processes have to be coordinated with visual or empirical evidence and mathematical results and facts, and are influenced by intuition and belief, by perceptions of authority and personal conviction, and by the social norms that regulate what is required to communicate a proof in any particular situation (see for example, Clements & Battista, (1992), Hoyles, (1997), Healy, & Hoyles, (2000).

The failure of traditional geometry teaching in schools stemmed at least partly from a lack of recognition of this complexity underlying proof: the standard practice was simply to present formal deductive proof (often in a ritualised two-column format) without regard to its function or how it might connect with students' intuitions of what might be a convincing argument: 'deductivity was not taught as reinvention, as Socrates did, but [that it] was imposed on the learner' (Freudenthal, 1973, p.402). Proving should be part of the problem solving process with students able to mix deduction and experiment, tinker with ideas, shift between representations, conduct thought experiments, sketch and transform diagrams. But what are the main obstacles to achieving this flexible habit of mind?

I present here some examples of geometrical questions that have turned out to be surprisingly difficult – even for high- attaining and motivated students. The analysis forms part of The Longitudinal Proof Project (Hoyles and Küchemann: http://www.ioe.ac.uk/proof/), which is analysing students' learning trajectories in mathematical reasoning over time. Data are collected through annual surveying of high-attaining students from randomly selected schools within nine geographically diverse English regions. Initially 3000 students (Year 8, age 13) from 63 schools were tested in 2000. The same students were tested again in the summer of 2001 using a new test that included some questions from the previous test together with some new or slightly modified questions. The same students will be tested again in June 2002 with the similar aims of testing understandings and development.

Question G1 in both Year 8 and Year 9 (see Fig 1), is concerned with how far students use geometrical reasoning to make decisions in geometry and how far they simply argue from the basis of perception or what 'it looks like' (see Lehrer and Chazan, 1998; Harel and Sowder, 1998). In both cases a geometric diagram is presented, which in the particular case shown, lends support to a conjecture that turns out to be false. Students are asked whether or not they agree with the conjecture and to explain their decision.

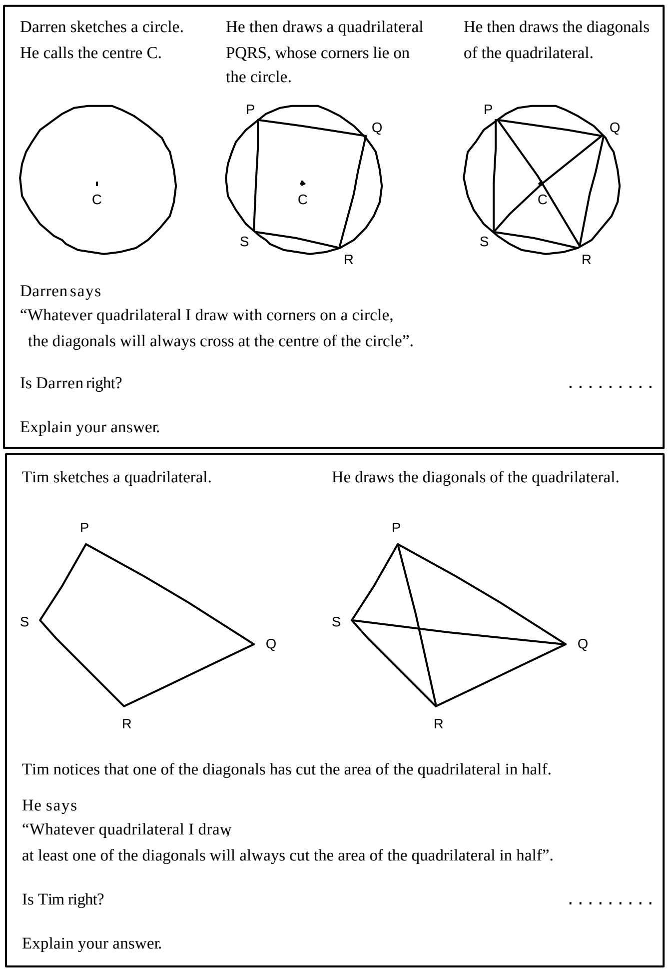

Figure 2: Question Y8G1 (top) and Y9G1: Distinguishing perceptual from geometrical reasoning

Responses to question G1 were coded into 6 broad categories. Surprisingly, a large number of students in both years simply answered on the basis of perception and agreed with the false conjecture with no evidence of progress over the year(Yr 8: 40 %, Yr 9: 48%). Additionally 41% in Yr 8 could come up with a correct answer and explain this by reference to an explicit counter-example while only 28% could do this in Yr 9.

Further analysis, however, is thought-provoking. Responses to Yr8G1 showed evidence of three effective strategies: the first to find the most extreme case that obviously shows that the diagonals cannot cross at the centre of the circle; the second to use dynamic reasoning, that is perturbate the diagram in an incremental way, keeping the given properties invariant (e.g., moving one of the vertices round the circumference so the intersection of the diagonals can no longer be at the centre), the third is to focus on the diagonals rather than the quadrilateral and simply to say 'I can find opposite vertices such that the diagonals do not go thru the centre' also evidenced by students who simply drew a diagonals 'cross' without bothering to draw the quadrilateral itself! In answer to Yr9G1, it is harder to find a counter example in a static way as two conditions have to be controlled (neither diagonal can bisect the area) rather than only one (one diagonal must not go thru centre); also it is not possible to find as 'extreme' a counter example as in Y8 (the nearest equivalent is a concave quadrilateral, though here it is still possible to end up with two triangles that look very different but have roughly the same area). The second strategy is also harder in the Year 9 question as the dynamic reasoning has to change an area, not an immediately obvious quantity unlike the coincidence of two points. Clearly avoiding the seduction of perception is only one pitfall in geometrical reasoning.

We also found that while students did not easily learn over time to reject perception, they did improve in calculation. Both Yr 8 and Yr 9 surveys included a question that required knowledge of certain angle facts (angle on a straight line or at a point, interior angle sum of triangle; angle property of isosceles triangle) and where a 3-step calculation had to be performed to find the size of an angle. We deliberately restricted the task to working with specific numerical values rather than asking students to derive a general relationship as would be required in a standard geometric proof, as this would simply be too hard for our students who have little experience of proving.

First, we note that students made considerable progress in their performance on the calculation part of the question (from 54% correin Yr 8 to 73% in Yr 9). But on analysing responses to the Yr 9 question, (see Figure 2), where we had asked students to give reasons for each step of their calculation, we discovered that not only did they find it hard to match a step in the calculation to a reason but also they were confused by *what it means to give a reason*. Many students interpreted ‘reasons’ in ways that we did not anticipate: that is, as an explanation for the step that they had taken(‘u is 40 as I took 40 from 360’), or as request to make their plans explicit (‘I started with p= 320 as the only thing that I know and I took it from 360 to find u’).

Our research is uncovering many more surprises in both student response and progress in proving - in geometry but also in algebra (One of our questions (for 14 year-olds) concerns the sum of odd numbers and shows a remarkably similar spread of responses as those described by Deborah Ball for children age 8/9 years. We know now even more about potential obstacles to ‘learning the mathematical game’; but need more systematic work on progress over time. there are no fool-proof approaches and no short cuts or easy solutions.

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4. The ‘Because for example…’ phenomenon, or transparent pseudo-proofs revisited

This panel is about the teaching of proof in mathematics, or as I interpret it - providing adequate conditions for gaining mathematics knowledge. My presentation is based upon the assumption that mathematics knowledge is *in principle* not different than any other kind of knowledge, although, of course, the nature of the discipline is different. What, then, is knowledge? According to Brook and Stainton

(2001), a common, long standing and most plausible answer, given by philosophers to this question, is that in order to be one's knowledge a proposition must comply with three necessary (albeit not sufficient) conditions:

(i) It must be true.
(ii) One must believe it. And
(iii) One must have justification for believing it.

Hanna and Jahnke (1993) suggest that in particular for a novice, a preliminary step towards appreciating what it is that is being justified, illumination - namely understanding and believing, is of maximum importance. Bertrand Russell makes an important distinction: Minds do not create truth or falsehood. They create beliefs. What makes a belief true is its correspondence to a *fact*, and this fact does not in any way involve the mind of the person who holds the belief. This correspondence ensures truth, and its absence entails falsehood. 'Hence we account simultaneously for the two facts that beliefs (a) depend on minds for their *existence*, (b) do not depend on minds for their *truth*….' (Russell, 1912).

We conclude that for a true mathematical statement, i.e., a theorem, to become one's mathematics knowledge, the learning environment must consist of teaching tools and strategies that support the development of two properties: (a) One's belief in its truth; and (b) one's ability to justify this belief, that is an ability not just to formally prove it, but also to ensure its truth by pointing out its correspondence to facts. Said differently, given a statement p of a mathematical theorem, a learner should be able to relate to two basic questions: (a) 'Do you believe that p?' and, provided the learner's answer to (a) is yes, (b)'Why do you believe that p?'

Quite often students' reply to the earlier question is of the form: 'Yes, because for example…'. Very seldom do the examples that follow, reflect full ability to verify the truth or even a partial understanding of it. For example, 'Yes, the sum of every two even integers is an even integer, because for example 6 plus 8 is 14', does not reflect any insight into the general case, although it does attest to an understanding of the statement, (which *cannot* be said about the reply: 'Yes, for example, because 14 is the sum of 6 and 8'!) The answer: 'Yes, because for example 6, which is 2x3, plus 8, which is 2x4, give 14, which is 2x7', is slightly better but not quite. It ties the belief to some acquaintance with the property of evenness. Although it may be based on deep understanding, it does *not* exhibit more than accepting the general claim as true, possibly due to a message from an external authority. (See also Mason 2001, about warrants and the origins of authority.) To be counted as 'satisfactory' the answer should be something like: 'Yes, because for example 6, which is 2x3, plus 8, which is 2x4, give 2x(3+4) and this IS an even number, as it is a multiple of 2' This latter one illustrates what we named *a transparent pseudo-proof.*

A transparent proof, is a proof of a particular case which is 'small enough to serve as a concrete example, yet large enough to be considered a non-specific representative of the general case. One can see the general proof through it because nothing specific to the case enters the proof.' Because a transparent proof is not a completely polished proof, this kind of 'proof' was later re-named *Transparent Pseudo-Proof* or as abbreviated: *Transparent P-Proof.* (Movshovitz-Hadar, 1988, 1998).

The delicate pedagogy involved in preparing a transparent p-proof was the focus of my ICME-8 Seville presentation (Movshovitz-Hadar 1998). That paper presents the lessons learned through experimental employment of two slightly different pseudo-proofs, both of them deserving the title 'P-Proof Without Words', yet only one of which - 'transparent'. The 1998 Samose presentation (ibid) included further insight into the notion of *transparent p-proof*, gained through the preparation of transparent p-proofs as pedagogical tools to be used in first year linear algebra course, at Technion - Israel Institute of Technology.

The study of the impact of using transparent p-proofs went on for four years, and yielded interesting results (Malek, in preparation). Numerous personal interviews of first-year mathematics majors and engineering students taking a linear algebra course, with exposure to transparent p-proofs, yielded clear evidence as to the impact of reading a transparent p-proof, on undergraduate students' ability to write, immediately afterwards, a formal proof of the same claim. A continuing follow-up also yielded comprehensive evidence as to the impact of reading transparent p-proofs, on the (passive) ability to read and comprehend general (formal) proofs, and most important of all, on the (active) ability to compose general proofs and write them in a coherent style.

Consequently, we now strongly advocate, wherever it is appropriate, the use of transparent p-proofs as a pedagogical tool, as it was shown to support both the development of one's belief in the truth of mathematical statements and of one's ability to justify this belief. However, it cannot be overemphasized that extreme care must be taken by the instructor in constructing this tool, be it in verbal-symbolic presentation or in visual-pictorial representation, so that the presentation is indeed of a *transparent* proof, namely, it does not hang in *any* way to the specifics of the particular case and hence is readily generalizable. The success of the resulting learning environment in yielding the development of the ability to prove, depends heavily on elaborate and careful preparation of the tools by the instructor.

Acknowledgement. The research work reported here was carried by Aliza Malek under my supervision, and was supported by Technion R&D funds.

5. Arguments from physics in mathematical proofs

Mathematicians often use arguments from physics in mathematical proofs. Some examples, such as the Dirichlet principle in the variational calculus or Archimedes' use of the law of the lever for determining the volumes of solids, have become famous, and have in fact been regarded by the best mathematicians as elegant proofs, if not necessarily rigorous. It is only natural, then, that several authors, notably Polya (1954) and Winter (1978), have proposed that arguments from physics could and should be used in teaching school mathematics. Besides these publications there are a number of other papers and booklets with examples (see, for example, Tokieda, 1998). Unfortunately, however, this approach to classroom teaching has not been sufficiently explored.

The application of physics under discussion goes well beyond the simple physical representation of mathematical concepts, and it is also distinct from drawing general mathematical conclusions by the exploration of a large number of instances. Rather, this approach amounts to using a principle of physics, such as the uniqueness of the centre of gravity, in a proof and treating it as if it were an axiom or a theorem of mathematics.

Let us look at a typical example. The so-called Varignon theorem states that, given an arbitrary quadrangle *ABCD*, the midpoints of its sides W, X, *Y*, *Z* form a parallelogram (see figure 3 below). A purely geometrical proof of this result would divide the quadrangle into two triangles and apply a similarity argument.

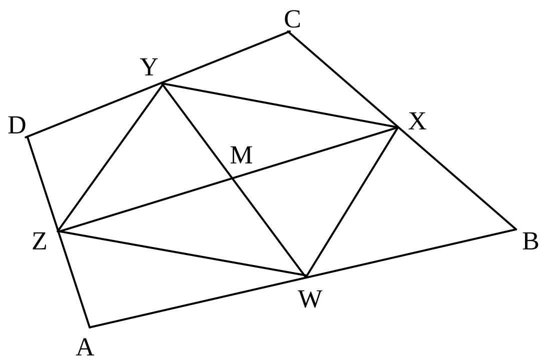

Figure 3: Varignon's theorem

An argument from mechanics, on the other hand, would consider points *A*, *B*, *C*, *D* as four weights, each of unit mass, connected by rigid but weightless rods. Such a system, with a total mass of 4, has a centre of gravity, and it is this which we need to determine. The two sub-systems *AB* and *CD* each have weight 2, and their respective centres of gravity are their midpoints *W* and *Y*. Thus, we may replace *AB* and *CD* by *W* and *Y* loaded with mass 2. Since *AB* and *CD* make up the whole system *ABCD*, its centre of gravity is the midpoint *M* of *WY*. In the same way we can consider *ABCD* as made up of *BC* and *DA* and its centre of gravity must also be the midpoint of *XZ*. Since the centre of gravity is unique, this midpoint must be *M*. This means that *M* cuts both *WY* and *XZ* into equal parts. Thus *WXYZ*, whose diagonals are *WY* and *XZ*, is a parallelogram.

The example shows that an argument from physics may

- provide a more elegant proof
- reveal the essential features of a complex mathematical structure
- point out more clearly the relevance of a theorem to other areas of mathematics or to other scientific disciplines
- help create a 'holistic' version of a proof, one that can be grasped in its entirety, as opposed to an elaborate mathematical argument hard to survey.

Frequently, an argument from physics helps to *generalize* and to arrive at new theorems. Following the lines of our previous argument, for example, we can determine the centre of gravity not only for systems with four masses, but also for those with three, five, six, and so forth. We can also consider three-dimensional configurations and investigate whether we are able to translate the respective statements about the centre of gravity into a purely geometrical theorem.

There are several reasons why this approach to the teaching of proof should be further developed and tested. First, it is unquestionable that, worldwide, we need fresh and possibly more attractive approaches to the teaching of proof. Since using arguments from physics in a proof is an alternative to the established Euclidean routine it might be helpful in motivating teachers to rethink their attitude to proof.

Another reason is that present-day mathematical practice displays a significant emphasis on experimentation, and it is only right that this be reflected in the classroom by a similar emphasis on experimental mathematics. But it would be dangerous from an educational point of view if experimental mathematics were to be represented in the schools only by 'mathematics with computers.' Quite to the contrary: under the heading of experimental mathematics, the curriculum should include a strong component devoted to the classical applications of mathematics to the physical world. In cultivating this type of mathematics, students and teachers should be guided by the question of how mathematics helps to explore and understand the world around us. In this way, the teaching of proof would be embedded in activities of building models, inventing arguments to the question 'why', the study of consequences from assumptions. Working on the border between mathematics and physics, it could be shown that in quite a few cases we cannot only apply mathematics to physics, but, vice versa, can use statements from physics for the derivation of mathematical theorems.

A Canadian and a German group (Gila Hanna, University of Toronto, Hans Niels Jahnke, Universität Essen) study the potentials and pitfalls of this approach in Canadian and German classrooms. Questions investigated concern the feasibility and the acceptance of the approach, given the limited knowledge of physics with students in both countries. It is also asked whether this approach furthers the general understanding of proof and whether the students are aware of the difference between using arguments from physics and the purely empirical appeal to a large number of instances (Hanna&Jahnke, 2002).

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