Proof and Justification in Collegiate Calculus

Manya Janaky Raman

B.A. University of Chicago 1992
M.A.T. University of Chicago 1993
M.A. University of California, Berkeley 1997

A dissertation submitted in partial satisfaction of the

requirements for the degree of

Doctor of Philosophy

in

Science and Mathematics Education

in the GRADUATE DIVISION

of the

UNIVERSITY OF CALIFORNIA, BERKELEY

Committee in charge:

Professor Alan Schoenfeld, Chair
Professor Andrea diSessa
Professor Alan Weinstein

Fall 2002

Proof and Justification in Collegiate Calculus

by

Manya Janaky Raman

Abstract

Proof and Justification in Collegiate Calculus

by

Manya Janaky Raman

Doctor of Philosophy in Science and Mathematics Education

University of California, Berkeley

Professor Alan Schoenfeld, Chair

This dissertation is a study of mathematical proof from both an empirical and a theoretical standpoint. The empirical component compares the views of proof held by entering university level students with those held by their two types of teachers: graduate student teaching assistants and mathematics faculty. The few studies that have been done at the university level focus almost exclusively on students, and most of those study populations of either preservice teachers or students in proof-based courses. Thus this direct comparison of views held by entering university students and their teachers provides needed data to help understand difficulties students face in making the transition from high school to university level mathematics.

The theoretical component, which is deeply intertwined with the empirical one, is to explicate the notion of proof. To do so, a distinction is made between a *private* and a *public* aspect of proof. The private aspect is that which engenders understanding and provides a sense of why a claim ought to be true. The public aspect is a formal argument with sufficient rigor (for the particular mathematical setting in which the argument is given) which gives a sense that the claim is true.

For the teachers in the study, the public and private aspects of proof are connected, through what is called the *key idea* of the proof. The key idea is the essence of the proof which gives a sense of why a claim is true and which can be rendered into formal rigorous argument. In contrast, for the students, the private and public aspects appear disconnected—in part because the students do not recognize the key idea of the proof, and in part because they do not even realize that the public and private aspects should be connected. It seems, then, that an emphasis on key ideas (including the understanding that they are eminently mathematical) may be an important mechanism for helping students develop a mature and epistemologically correct view of mathematical proof.

To my parents, for my education

Table of Contents

Acknowledgements

I have received help from many people in conceiving, creating, and finally completing this dissertation. First, my deep thanks go to my advisor, Alan Schoenfeld, for all his support throughout this project. There were times, especially near the end, when I relied heavily on his encouragement and constructive criticism to move forward with my work. Thanks also to my other committee members, Alan Weinstein and Andy diSessa. Alan gave useful insights for the design of the project from his many years of teaching calculus. Andy's comments were always thought provoking and pushed for a level clarity I wish I could have achieved with this current work.

Various versions of this work have been presented to the Functions Group at UC Berkeley and the Mathematics Didactics Group at Umeå University. I thank members of both groups for helpful comments and support along the way, especially my Swedish hosts Johan Lithner and Hans Wallin. Thanks to Kate Capps and Melinda Schissel for the innumerable headaches from which they saved me as I split my time between Berkeley and Sweden during my graduate years. And thanks to Alf Jonsson and the rest of the Umeå mathematics department for their warm welcome and use of facilities.

Earlier versions of this work were also presented at national and international conferences as well as invited seminars. I benefited from criticisms and comments from people who attended those talks. I especially want to thank Guershon Harel, Eric Knuth, Michelle Zandieh, and Anna Sfard. Recent collaboration with Margaret Morrow, who has conducted a duplication of this study at another site, has been a boon. Margaret provided useful comments on Chapter 1 and also helped with a few references.

I am also grateful to the Spencer Foundation for funding during the final year of the project. One benefit of this support was a chance to present my work to members of my cohort and invited scholars. I thank all the members of my group for their feedback on an earlier version of this work, and a special thanks to Lance Ripps for, among other things, taking me seriously.

I am also indebted to the twenty participants who took part in this study. Their participation was entirely voluntary, and in some cases involved several hours above and beyond the minimum commitment I had asked. It was very gratifying to find out that many of the participants wanted to participate because they believed in the usefulness of the project.

Finally, thanks to my family and friends for helping me make it through to the end. Carl, Beth, Lloyd, Sharon, Gunnel and Tage, you have been my homes away from home. Monica, you are sent from heaven. And Pär, you have my deepest thanks and love for keeping me rooted to the earth.

Chapter 1: Introduction

Since the 6th century BC when Greek mathematicians established the axiomatic method, mathematicians have considered proof to be the *sine qua non* of mathematics. As Dreyfus asserts, "Proving is one of the central characteristics of mathematical behavior and probably the one that most clearly distinguishes mathematical behavior from behavior in other disciplines." (Dreyfus 1990, p. 126)

However, the question of what proof really *is* has been, and continues to be, a matter of debate among mathematicians, historians of mathematics, philosophers of mathematics, and mathematics educators. The following caricature of an ideal mathematician, from *The Mathematical Experience* by Davis and Hersh, illustrates part of the debate and some of its implications for teaching.

A student visits the ideal mathematician and asks him what a proof is. The mathematician responds that a proof is a transformation from a hypothesis to a conclusion using the rules of logic, based on the axioms and the symbols of a formal system. The student is surprised, saying that she has never seen a proof like that in her undergraduate mathematics courses. She pushes the mathematician to explain further. Reluctantly the mathematician gives in and says, "Well, it's an argument that convinces someone who knows the subject." The student points out that that implies proof is subjective. The mathematician stubbornly objects, "No, no! There is nothing subjective about it! Everybody knows what a proof is. Just read some books, take courses from a competent mathematician, and you'll catch on." (Davis and Hersh, 1981, pp. 39-41)

Unfortunately it appears that most students do not simply "catch on." Research indicates that students both at high school and university level have difficulty, not only in producing proofs, but even in recognizing what a proof is (Galbraith, 1981; Fishbein and Kedem, 1982; Vinner, 1983; Chazan, 1993; Moore 1994). Difficulty in understanding the nature of proof has also been reported among prospective elementary school teachers (Goetting, 1995; Martin and Harel, 1989; Simon and Blume, 1996) and experienced high school teachers (Knuth, in press). Understanding the nature of proof, in addition to being of theoretical interest, seems essential for thinking about how to teach students about proof, both at the university level and throughout the K-12 level, as suggested by the NCTM standards (NCTM, 2000).

This dissertation is a study of mathematical proof from both an empirical and a theoretical standpoint. The empirical task is to compare the views of proof held by entering university level students with those held by their two types of teachers: graduate student teaching assistants and mathematics faculty. The claim has been made that university student and teacher views conflict (e.g. Alibert and Thomas, 1991; Harel and Sowder, 1998) but there has been little research documenting the nature of the conflict. Because it looks at students who are just beginning collegiate mathematics, whose views are formed in large part by the views of proof they have developed in high school, the study also sheds some light on differences between high school and university level treatments of proof.

The theoretical task, which is central to completing the empirical one, is to explicate the notion of proof. To do so, I distinguish between a *private* and a *public* aspect of proof, the private being that which engenders understanding and provides a sense of why a claim is true. The public aspect is the formal argument with sufficient rigor for a particular mathematical setting which gives a sense that the claim is true. This distinction has some precedents in the literature, as discussed below.

For the teachers in the study, the public and private aspects of proof are connected, through what I call the *key idea* of the proof. The key idea is the essence of the proof which gives a sense of why a claim is true and can be rendered into formal rigorous argument. In contrast, for the students, the private and public aspects appear disconnected, in part because the students do not recognize the key idea of the proof, and in part because they do not even realize that the public and private aspects should be connected. It seems, then, that an emphasis on key ideas (including the fact that they are eminently mathematical) is important for helping students develop a mature and epistemologically correct view of mathematical proof.

The notion of proof

The literature on proof is vast, and comes from many different sources: philosophy of mathematics, history of mathematics, and mathematics education, to name a few. People from different disciplines approach the question of defining proof with different agendas in mind. Many philosophers, for example, want to understand proof in terms of the fundamental nature of mathematics. In contrast, many educational researchers are more interested in students' or teachers' understandings of proof. However one message from this vast literature is clear: proof has many, seemingly contradictory, aspects. One is logical, as the following definition indicates.

> Definition 1 (Logical): A finite sequence of propositions each of which is either an axiom or follows from preceding propositions by one of the rules of logical inference. (The Columbia Encyclopedia, 2001)

This view of proof, touted by the ideal mathematician above, is also commonly expressed in mathematics textbooks. Take for instance this quote from an "Introduction to Proof" text written to help university level students understand the nature of mathematical proof.

> In mathematics things are proved; in other subjects they are not. […] The ancient Greeks found that in arithmetic and geometry it was possible to prove that results were true. They found that some truths in mathematics were obvious and that many of the others could be shown to follow logically from the obvious ones. (Franklin and Daoud, 1988, p.1)

This view of proof has been challenged by many people, perhaps most notably philosopher Imre Lakatos, who argues that Definition 1 refers only to a narrow class of proofs within a formal theoretical system (Lakatos, 1978; see also Schoenfeld, 1994). Another kind of proof, used by everyday mathematicians in everyday situations, is what Lakatos calls informal or pre-formal proofs. Pre-formal proofs can be completely convincing but not proofs in a logical sense. So proof has not only a logical nature, but also a psychological one. The idea of proof as a convincing argument, as shown in Definition 2 below, is also fairly common, if not in formal texts, at least informally among mathematicians and educators.

> Definition 2 (Psychological): An argument one gives in order to convince others (and often one's self) of the correctness of one's assertions. (Movshovitz-Hadar, 2001)

The psychological and logical aspects of proof are often pitted against each other, as the following quote indicates.

> The main purpose of a *logical* approach is to convince doubters; that of a *psychological* one is to bring about understanding. (Skemp, 1971, p. 13)

This distinction makes some sense. One can follow, or even create, a logical proof without understanding it. A case in point is Deligne, a Field's medalist, who after producing a formal proof of a difficult theorem, said, "I would be grateful if anyone who has understood this demonstration would explain it to me." (Alibert and Thomas, 1991, p. 220) Similarly one can believe something is true without being able to produce a formal proof. For instance, it may appear intuitively obvious that if a continuous function is positive for some value of x and negative for another, it must be zero at some point in between. However, as history has shown, coming up with a rigorous proof of this claim is far from trivial.

However, it seems wrong to separate the logical and psychological aspects of proof completely, at least the way Skemp has done. Skemp has made it seem like convincing and understanding are two separate activities. It is not clear a priori why this should be true. Understanding the truth of a claim at a psychological level may very much have to do with developing a sense of conviction. Producing a logical proof may very well help develop understanding. Thus it seems less than clear how the psychological and logical aspects of proof are related, and whether or not they capture all of what proof entails.

The answer to the latter question appears to be that they do not. In addition to a logical and psychological nature, recent research indicates that proof has a social nature as well. An increasingly popular view is that mathematical proof is a human product and humans are fallible. For instance as Hersh asserts in a recent forray into the philosophy of mathematics, "mathematics must be understood as a human activity, a social phenomenon, part of human culture, historically evolved, and intelligible only in a social context." (Hersh, 1997, p. xi)

Even if we take proof to be a convincing argument we can ask, what is convincing, and to whom, and in what situation? Standards for proof have varied across time. "Euclid would probably have complained of the lack of rigor displayed by his predecessors; Weierstrass felt it necessary to reorganize the foundations of mathematics." (Wilder, 1944, p. 315, see also Grabiner 1998 and Kleiner 1991) Standards for proof also vary across discipline domains. Each disipline is a mathematical community with certain sets of shared assumptions. An argument that could be expressed easily to a group in one field might need a lot more background to be expressed to another group. One case in point is Thurston, a mathematician, trying to explain one of his proofs to both toplogists and analysts. The way he communicated his ideas and the amount of time varied dramatically between the two groups. He concludes, "It became dramatically clear how much proofs depend on the audience. We prove things in a social context and address them to a certain audience." (Thurston, 1994, p. 175)

Changes in the standards for proof have been especially notable in the area of analysis, of interest here because we study proof views in the context of collegiate calculus. The way we currently define the central concepts in collegiate calculus—limits, derivative, integral—came about from a need to arithmetize these notions, that is to free them of their geometric and intuitive aspects. When calculus was conceived by Newton and Leibniz, numbers were thought of geometrically. "Dedekind's and Weirstrass's astute insight recognized that a rigorous arithmetical definition of the real numbers would resolve the major obstacle in supplying a rigorous foundation for the calculus." (Kleiner 1991, p. 301, see also Kline, 1972, pp 947-978).

The effects of the rigorization of calculus can be seen in modern textbooks and classrooms. Instead of proving the intermediate theorem using a geometrical argument (look, the curve intersects the x-axis), we use an analytic argument using the least upper bound property of the real numbers. However, one can still ask wherein the proof lies. Brown (1997) argues that the visual evidence is completely convincing. The analytic argument simply confirms the premisses of a claim already known to be true. According to Brown then, pictures can prove. He goes against the mainstream to defend this thesis:

> Though not universal, the prevailing attitude is that pictures are really no more than heuristic devices; they are psychologically suggestive, and pedagogically important, but they prove nothing. I was to oppose this view and to make a case for pictures having a legitimate role to play as evidence and justification, well beyond a heruistic role. In short, pictures can prove things. (Brown, 1997, p. 161)

(This view, by the way, was held by several mathematicians in the study reported here and none of the students.)

Even if we consider one mathematician in one specific domain working on the proof of one particular theorem, the standards for proof are less clear than we might expect. Consider this description by deMillo et al of what happens in practice:

> A proof is not a beautiful abstract object with an independent existence. No mathematician grasps a proof, sits back, and sighs happily at the knowledge that he can now be certain of the truth of his theorem. He runs out into the hall and looks for someone to listen to it. He bursts into a colleague's office and commandeers the blackboard. He throws aside his scheduled topic and regales a seminar with his new idea. He drags his graduate students away from their dissertations to listen. He gets onto the phone and tells his colleagues in Texas and Toronto. In its first incarnation, a proof is a spoken message, or at most a sketch on a chalkboard or a paper napkin. (deMillo, 1998, p. 272)

De Millo goes on to explain other stages that the proof goes through before it is really believed. If the proof generates excitement, the professor would write it up and circulate it to colleagues for feedback. Then begins the rigorous process of peer review for a journal. As more and more people accept it, the proof gains credibility. So one might be led to believe that in practice, a proof is an argument which has passed through enough hoops to be considered believable.

Things get even messier when one considers computer-aided proofs, like the proof of the Four Color Theorem, which are too long to be checked by humans. This type of proof led Devlin to give us our third characterization of proof:

> A proof is often just an argument that
> (i) has been accepted by a number of mathematical whom the community at large feels it can trust on such matters, and
> (ii) has not yet been shown to be false. (Devlin, 1995)

As a product of human invention, the nature of proof becomes much more fuzzy than the logical definition would lead us to believe it to be.

Public and private aspects of proof

Given that proof has logical, pychological, and social aspects, providing a characterization of proof is not an easy task. Recent work has attempted to create some order by making various distinctions among different types of proofs. For instance as Steiner (1978) points out and Hanna (1989, 1990) emphasizes, mathematicians routinely distinguish between proofs that demonstrate and proofs that explain (though it remains an open question what it means for proofs to explain).

> A proof that proves shows only that a theorem is true; it provides evidential reasons alone. It is concerned only with substantiation, with what are known as *Rationes cognoscendi*, that is why-we-hold-it-to-be-so reasons. A proof that explains, on the other hand, also shows why a theorem is true; it provides a set of reasons that derive from the

> phenomenon itself: *Rationes essendi*, or why-it-is-so reasons. (Hanna, 1990, p. 9)

Proof has also been described as a part of a discourse practice (Sfard, 2000; Sfard, private communication), with a distinction being made between discourse with oneself (in the case of trying to produce a proof) and discourse with others (in the case of trying to communicate a proof). This distinction is similar to that made by Harel and Sowder (1998) between the process of ascertaining ("removing one's own doubt"), and persuading ("removing other's doubt"). It also bears resemblance to Mason's (1985) pedagogical suggestion of how to create a proof: first convince yourself, then convince a friend, then convince an enemy.

In all these cases, people seem to distinguish between an essentially *private* and an essentially *public* aspect of mathematics; that is to say proof involves both public and private arguments. By "private argument" I mean, "an argument which engenders understanding", and by "public" I mean "an argument with sufficient rigor for a particular mathematical community". "Mathematical community" refers not only to a particular setting, but also the people involved along with their expectations for the kind of argument needed within that setting. Examples of mathematical communities could include formal settings like an exam, publication in a journal, or an informal setting like office hours or a conversation between two mathematicians in the same field. One can also distinguish between public and private senses of conviction, the public sense being based on public arguments and the private sense being based on private arguments.

The argument that I will advance in this thesis is that for mathematicians working on a fairly standard calculus proof, the private and public aspects of proof are linked, through what I will refer to as the "key idea" of a proof, while for the students the public and private aspects are decidedly separate. I will distinguish the key idea from two other kinds of ideas, one heuristic, which is intuitive and could lead to a proof but has not, and the other procedural, which could lead to a proof, but does not by itself engender understanding. The key idea is one that contains the essential idea of the proof and can be mapped to an argument with the appropriate level of rigor for the context. While an intuitive idea gives a sense that a claim *ought* to be true and a procedural idea demonstrates *that* a claim is true, the key idea provides a sense of *why* a claim is true.

"Key Idea" in the literature

The notion of "key idea," while does not discussed as such, has precedents in the literature. For an example we turn again to mathematician Thurston who explains how he reads a mathematical paper in a field in which he is conversant:

> I might look over several paragraphs or strings of equations and think to myself, "Oh yeah, they're putting in enough rigamarole to carry such-and-such idea." When the idea is clear, the formal setup is usually unnecessary and redundant—I often feel that I could write it out myself more easily than figuring out what the authors actually wrote. (Thurston, 1994, p. 167)

To the mathematician, what is important about the proof is the idea it expresses. The symbols and formalism used to express that idea are just 'rigamarole' for carrying that idea. As Hanna argues, "mathematicians, including those who have recourse to purely syntactic methods, are really more interested in the message behind the proof than in its syntax, and see the mechanics of proof as a necessary but ultimately less significant aspect of mathematics." (Hanna, 1990, p. 12; See also Hanna 1983)

Even Bourbaki, a group of French mathematicians known for their "uncompromising adherence to the axiomatic approach" (Boyer, 1989, p. 706), seems to agree with this point of view:

> Indeed every mathematician knows that a proof has not been "understood" if one has done nothing more than verify step by step the correctness of the deductions of which it is composed and has not tried to gain a clear insight into the ideas which have led to the construction of this particular chain of deductions in preference to every other one. (Bourbaki, 1950)

This view of proof seems in stark contrast with students who see the formalism as a central part of the proof. One case in point are two geometry students working on a construction problem (Schoenfeld, 1985, p. 367). The students come up with a solution that convinces them, but then since their solution does not involve the proper form they have been taught in school (in a T diagram with statements on the left and reasons on the right), they do not think the work they have done is mathematical.

> TH: So we can bisect this to find the center, right? So call it center C. [she pauses] Maybe we should have done our steps. [emphasis added]
> LS: [Referring to the way they had proceeded on the conjecture up to that point] That's all being unmathematical, completely disorganized.

The students do not see the informal work they have done as grounds for the proof. The 'steps' of a proof, rather than being a way of systematizing their ideas seems something separate from the (in this case correct) underlying ideas.

Balacheff (1988) has also found that students have difficulty making connections between informal work, based on examples, and a formal looking proof. In one study, students are asked to find a general expression for the number of diagonals in a polygon. A pair of students consider the case of a hexagon and realize that the lines between adjacent vertices won't count as diagonals. They write:

> In a polygon if there are 6 points there will automatically be 3 diagonals for each point because in the boundary of the polygon there are two points which join it on: conclusion there are 3 which are diagonals. (Balacheff, 1988, p. 226)

The students then know that they want to produce a 'general' proof. They say, "There we've done an example…you musn't use examples… it must be general." However, the way they see of generalizing the argument above is to simply replace *x* for 6 and *y* for 3. They do not see that the key idea is that each vertex will connect to *n*-3 vertices of the *n*-gon (with each diagonal thus being counted twice). The need to produce a formal answer for the sake of being formal leads the students to create a meaningless sentence with nowhere else to go.

In recent years, research on proof has shifted from documenting students' difficulties with proof, to accounting for students' abilities and perceptions about proof in different types of settings. For instance Hoyles and Healy try to tease apart students' views of what counts as a proof from what they produce when left to their own devices (Hoyles and Healy, 1999). In a study of 2,459 high achieving students in grade 10 in England and Wales, Hoyles and Healy found that students thought that a formal-looking proof would get highest marks on an exam, but that they personally preferred a narrative or empirical approach. Similarly in the domain of physics, Elby (1999) got substantially different results when he asked students what kind of argument students would give in a class setting and what kind of argument they would give to help themselves understand without time and grade pressure.

These studies seem to demonstrate the existence of both private and public arguments. The study reported here builds on these studies, not only in accounting for differences between the public and private domains, but trying to understand how these domains are viewed by students and teachers at the university level.

Organization

The remaining chapters in this dissertation describe a study in which the main goal is to characterize the views of proof held by entering university level students and their teachers. As stated above, a direct comparison of this kind has not been done before, thus this dissertation provides needed data on what is known to be a difficult transition between high school and university level mathematics.

In order to capture the differences between student and teacher views, a method was used, adapted from Hoyles and Healy (1999), which involves producing and evaluating proofs in different contexts. We study both how people attempt to produce a proof on their own with how they evaluate different ready-made proofs according to different criteria. The criteria used were: which is most convincing? which demonstrates best understanding? which would get highest marks on an exam? which do you prefer? These methods are described in more detail in chapter 2.

The analysis of the data is divided into two chapters—chapter 3 focuses on how participants attempted to produce a proof on their own, and chapter 4 focuses on how they evaluated to five ready-made proof (or proof-like) responses. In chapter 3 the analysis focuses on the public and private arguments employed to create (or attempt to create) a proof and whether the participant is aware of connections between them. In chapter 4, in which the burden of producing the proof is removed, the data and analysis provide a more articulated view of the differences between how students and teachers view the public and private arguments and supplies additional data supporting the theoretical notion of key idea.

Chapter 5, more speculative than the previous chapters, draws on data from the study as a springboard for discussing some of the implications of the study for teaching. In light of the study we would like to know how key ideas could be more emphasized, both in teaching and testing. While there is no magic bullet, we discuss some of the possible avenues for implementation, including oral examinations and more complex mathematical tasks.

Chapter 2: Methods

The aim of the study was to compare the views of proof held by entering college freshmen and their two types of teachers, graduate students in mathematics and mathematics faculty. The method used was adapted from Hoyles and Healy (1999), who compared the proof views of high school students and their teachers. The main difference was that the study here focused on a smaller number of students with more in-depth interviews. The data consists mostly of task-based interviews centered around the task: *Prove that the derivative of an even function is odd*. This is a task that appears in most calculus books. While it is fairly difficult for the average student, the central notions involved—derivative, even, and odd—are ones that are familiar to them. Another nice feature of the task, as was the case with the tasks used in Hoyles and Healy (1999), is that it can be approached in various ways (five of which are given below). This allowed for discussion with each participant about how they viewed different responses according to different criteria, also discussed below.

Participants

The site chosen for the study was a top-ranked university in the United States. There were twenty participants in the study: 11 freshmen calculus students, 4 mathematics graduate students, and 5 mathematics professors, each of whom were interviewed individually. Profiles of the participants are given in Tables 2.1, 2.2, and 2.3 below.

Professor	Field	Gender	Years teaching calculus
Prof. A	Mathematical Economics	Male	Moderate
Prof. B	Differential Equations	Male	Extensive
Prof. C	Geometry	Male	Extensive
Prof. D	Numerical Analysis	Male	Extensive
Prof. E	Complexity Theory	Male	None

Table 2.1: Professors in Study[1]

Graduate Student	Field	Gender	Year in Grad School	Years as calculus teaching assistant
Grad. A	Operator Algebras	Male	Beginning	Little
Grad. B	Dynamics	Female	Beginning	Little
Grad. C	Topology	Male	Middle	Moderate (but first time in US)
Grad. D	Analysis	Female	Advanced	High

Table 2.2: Graduate Students in Study[2]

[1] Teaching Experience: Little (1- 2 semesters), Moderate (3 – 8 semesters), High (8-12 semesters), Extensive (more than 12 semesters).
[2] Year in grad school: Beginning (1-2 years, before orals), Middle (3-4 years, after orals, just starting research), Advanced (5+ years, about to finish).

Student	Major	Gender	High school math experience	College math experience
Student A	unknown	Male	Took BC Calculus, got 3 on AP exam[3]	Skipped to 2nd semester calculus, got a D, got discouraged. Now retaking 1st semester calculus.
Student B	unknown	Female	Took calculus at college level during junior year of high school.	Finds current college course very challenging, but thinks tests do not test understanding.
Student C	unknown	Male	Got A in high school calculus, 3 on AP test	High school math was a 'joke'. Never did proofs before college.
Student D	Computer Science	Male	Got A's in the Interactive Math Program curriculum.	Math was favorite class in high school, but not in college. Not getting A's now.
Student E	Some area of science	Female	Thought she was a 'math person'	Taking calculus 2nd time, getting B. No longer thinks she is a 'math person'.
Student F	Nuclear Engineering	Male	Took AP calculus junior year, 5 on AP exam.	Getting A-.Difference between high school and college: "Here he wants us to be able to manipulate what we know, not just do things like we do in homework problems."
Student G	Economics	Female	Got As in math	Doing about average in calculus. Says college differs from high school because you are expected to know 'why'. She doesn't think she needs to know why because she is going into business.

[3] The Advanced Placement (AP) exam allows students to get college credit for calculus while in high school. The AB exam covers the content of first semester calculus and the BC exam covers the content of first and second semester calculus.

Student	**Major**	**Gender**	**High school math experience**	**College math experience**
Student H	Molecular and Cell Biology	Male	Moved to US for high school. Got As in math until calculus. Got B in calculus.	Grades are worse in college calculus. "High school is basically memorizing formulas. Here in college you have to know exactly how to find, how to come up with that formula."
Student J	Pre-med	Female	Took AB calculus got As in math. Got 4 on AP exam. Felt she understood math until 10th grade.	Got 52/100 on second midterm. Above mean, but still demoralizing for her.
Student K	Business	Female	Took AB calculus. Math in Burma until 8th grade. Math was much easier in USA.	Gets A on exams and quizzes.
Student L	Computer Science	Male	Took honors classes in high school, but didn't do well. 3 on AP exam. Math in Russia until 6th grade. Math easier in US.	Says he is not doing well in college calculus, but he likes mathematics.

Table 2.3: Undergraduate students in study

The students, all of whom were college freshmen, came from three sections of a first semester calculus course for intended science and engineering majors. One of the courses was taught by Prof. A, and as a second semester course, contained many students who were taking college calculus for the second time. Grad. A was Prof. A's teaching assistant, and his students were Student A, Student B, Student C, Student D, and Student E. The second course was taught by Prof. B, and was a first semester course. Grad. B and Grad. C were Prof. B's teaching assistants. Grad. B's students were Student F, Student G, and Student H. Grad. C's students were Student J, Student K, and Student L. The other professors and graduate student were chosen to get a wider sample of teachers. The tree diagrams below, Figure 2.1, shows which student attended which section of each course, for easy reference.

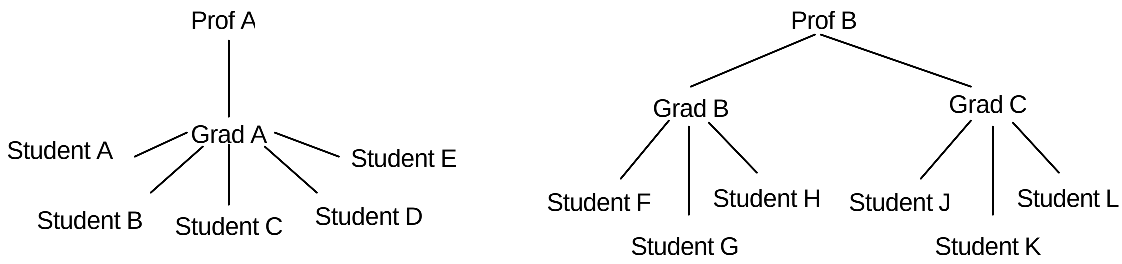


Figure 2.1: Relationship between participants

The population was diverse in terms of their country of origin. Teachers hailed from: Denmark, China, India, New Zealand, Russia, and USA. Students came from Kenya, Burma, Taiwan, Egypt, Bosnia, Russia, and USA, though all attended at least high school in the United States.

Selection was made in part on a volunteer basis. The two calculus courses happened to be the courses running during the time the study was to be conducted. The teaching assistants were the first ones contacted who agreed to be part of the study. The interviewer, myself, attended the sections of these teaching assistants for several weeks. At the beginning of my visit, I collected a list of volunteers among the students. Based on my own observations and discussions with the teacher, students were chosen from this list to represent a wide variety of ability and success in the calculus course. A few students who had not originally volunteered were then asked to join the sample, and they did so willingly.

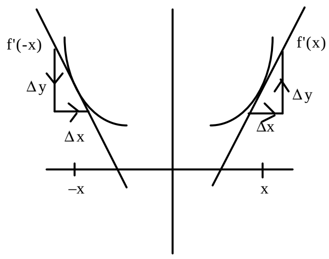
Setting

The interview consisted of three parts: background questions, a task, and response comparison. Each interview lasted 1-1.5 hours and was both audio and videotaped. The main purpose of the background questions, in addition to getting some contextual data, was to make people feel at ease in front of the camera and to set a relaxed tone for the remainder of the interview. Students were asked about their high school and college mathematics experiences, as well as information such as year, major, and grades.

Teachers were asked about teaching experiences, their area of study, and about any shifts they have had in their view of mathematics over the years.

The second part of the interview centered around the following task: *prove that the derivative of an even function is odd.* Participants were first asked to answer this question on their own. They were given time to work on the question silently and then were asked to explain their work, focusing on why they chose a particular method and how convinced they were of their response.

Next the participants were shown five different responses to the question and asked to evaluate them. The responses, not all of which were correct, came from pilot work. These responses are given below. Note that if one is being very careful one should specify that the function is differentiable. Several professors commented on this, but the task was given just as it is in the textbook used in the course (Stewart, 1998).

Response 1:	Response 2:
Consider the following functions and their derivatives -- f(x) = x odd f(x) = x2 even f'(x) = 1 even f'(x) = 2x odd f(x) = x3 odd f(x) = x4 even f'(x) = 3x2 even (x) = 4x3 odd f(x) = x5 odd f(x) = x6 even f'(x) = 5x4 even f'(x) = 6x5 odd Note that for all even functions, the derivative is odd. We could continue for all powers (n = 7, 8, 9, ...), thus the claim is proved.	 If f(x) is an even function it is symmetric over the y axis. So the slope at any point x is the opposite of the slope at -x. In other words f'(-x) = - f'(x), which means the derivative of the function is odd.

Response 3:	Response 4:
Want to show if f(x) = f(-x) then - f'(x) = f'(-x). f'(-x) = $\lim_{h\to 0}\frac{f(-x+h)-f(-x)}{h}$ by the definition of derivative $=\lim_{h\to 0}\frac{f(x-h)-f(x)}{h}$ since f is even Let t=–h $=\lim_{t\to 0}\frac{f(x+t)-f(x)}{-t}$ $=-\lim_{t\to 0}\frac{f(x+t)-f(x)}{t}$ = –f'(x) as desired.	Given f(x) is even, so f(x) = f(-x). Take the derivative of both sides. f'(x) = -f'(-x) by the chain rule. So f'(x) is odd.
Response 5: f is even, so f(x) = f (-x). Multiply both sides by -1 - f(x) = - f(-x) Factor in -1 f(-x) = - f(-x)	Response 5, cont. f is even so we can substitute f(-x) = f(x) f(-x) = - f(x) Take the derivative of both sides f'(-x) = - f'(x) So f' is odd, as required.

Participants were asked to judge each response based on different criteria:

(1) Is it convincing? Why or why not?,

(2) How many points would this get on an exam? Why?

(3) What response (or parts or combinations of responses) do you prefer? Why?, and

(4) What response (or parts or combinations of responses) would demonstrate the best understanding? Why?

Triangulation techniques

A second round of interviews was conducted to address issues of validity and scope that arose after a preliminary analysis of the first round data: how reliable are the professed beliefs, how typical are the beliefs/understandings of the individuals in the study, and how well do the five responses shown to people in the study provide access to people's beliefs and understandings about proof? A sample of participants from the first round (1 professor, 1 graduate student, and 2 undergraduates) was chosen from those who volunteered to be interviewed again. The participants were interviewed one semester after they were originally interviewed. They were shown some of the results of the study (which included comments from un-named participants in each group about each of the responses, see Appendix A) and were asked whether the views expressed in those comments seemed typical of view of people in each of the three groups, and whether the views expressed in those comments reflected his or her own personal views.

The data were then analyzed and a paper was written presenting the results (see Appendix B). This paper was then sent to all the participants in the study, and comments were solicited. Comments were received from 1 student, 2 graduate students, and 1 faculty member. In addition, several participants were contacted by email to clarify aspects of their interview before this final report was written.

Chapter 3: Data and analysis, part 1

In this chapter and the next we report the empirical results of the study. This chapter focuses on what the undergraduate students, graduate students, and faculty did as they attempted to prove that the derivative of an even function is odd. Perhaps unsurprisingly, the undergraduate students found it difficult to prove the claim and no student came up with a completely accurate proof. The graduate students and mathematics faculty were able to dispense with the problem fairly quickly.

In looking at the different ways that students and faculty attempted to prove the claim, we find that there are three distinctly different types of approaches. One, based on what I call heuristic ideas, gives a sense that the claim ought to be true, but does not necessarily lead to a proof. Another, based on what I call procedural ideas, demonstrates that the claim is true, but does not necessarily engender understanding. The third, based on what I call the key idea, demonstrates why the claim is true, providing both a sense of conviction and understanding. The key idea links what I call the private arguments, those which engender understanding, and the public arguments, those which are expressed in language of sufficient rigor for the situation.

The main difference between the faculty and the students is that the faculty think of proof in terms of the key idea—they either prove the claim using a key idea or they recognize that they could—while the students see the private and public arguments as decidedly separate.

Proof production

Faculty

All of the faculty in the study generated correct proofs of the claim that the derivative of an even function is odd. There were two different types of approaches. Either (1) the professor thought about examples and/or pictures to convince himself first that the claim is true and then produced a formal argument using the chain rule or the definition of derivative, or (2) he produced a formal argument directly. Below we consider an example of each type.

Professor B begins by thinking about the even functions x^2 and cosine x . He

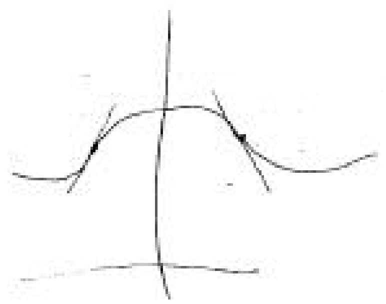


Figure 3.1: Prof. A's sketch of an even function

sketches what he says is a generalized graph of an even function, which looks like part of a cosine function. He draws the tangent line at a point to the right of the axis and one to the left of the axis as indicated in Figure 3.1. Then he reasons as follows:

> Prof. B: If it is even then it has to be the same to the left and to the right. If it is the same to the left and to the right then clearly, if you draw the tangent line, that is reversed when you flip across the y axis. The function is preserved because it is even, but the slope is reversed.

Prof. B then says he would need to stop and think if he "needs to say anymore". As he thinks outloud he stumbles on a claim that turns out not to be true (he claims that a constant function is both even and odd) and then wants to go back and check if his proof is right.

> Prof. B: It's always good to go back and see if your initial response to something is right. Let's see, an even function. There is only one thing about it, and that is its graph is reflected across the axis. Yeah, and you can be quite convinced that it is true by looking at the picture. If you said enough words about the picture, you'd have a proof.

He then goes on to write an argument using the chain rule.

> Prof. B: Then you could say, well, suppose you want to write it out in formulas. You could say that means, (writes) f of x is equal to f at minus x. And then you could say, if it's differentiable, you could differentiate both sides. And then you'd say, well ok that means f prime at x is equal to negative f prime at negative x by the chain rule. This is odd, the f prime function.

Professor B has now generated two different arguments for why the derivative of an even function is odd. Wherein does the proof lie? It appears that the professor wants to say two seemingly contradictory things. When he first generated the picture he says clearly that the picture is not a proof. The picture convinces him that the claim is true, but you would need to "say enough words" about it to make it a proof. However, later in the interview he makes the stronger claim.

> Prof. B: Pictures can prove things, that's what convinces you.

On the one hand pictures prove. On the other hand, you have to say more words about a picture to make an acceptable proof.

Thus, proof for Prof. B has both a *private* and a *public* aspect. The *private* aspect is what is personally convincing. It is an insight or idea which resonates with personal experience in a way that gives a sense of why the claim ought to be true. Prof. B started with hands-on examples, abstracted that to a picture which captured for him the essence of an even function (its symmetry across the y axis), and then made an observation about that picture which convinced himself of the truth of the claim. The *public* aspect is the formal argument, with appropriate rigor for the situation, a translation of some informal idea or argument into the coin-of-the realm. Professor B is not satisfied with his picture proof. He feels he either needs to say more words about it to make it an acceptable proof, or come up with an argument that doesn't involve pictures at all ("Then you could say, well, suppose you want to write it out in formulas. You could say that means, (writes) f of x is equal to f at minus x"). Note the use of the word "means". For the professor, the publicly acceptable formula is a translation of his privately held idea into the publicly accepted coin-of-the-realm.

We can see the distinction between the public and private aspects of proof clearly when the professor is asked to articulate the difference between his picture argument and the chain rule argument.

> Prof. B: Oh, the first one convinces me completely that it is right, it is right. The second one is how you present it if you want to convince somebody else. It's not… it doesn't have… (sigh, look to side) your currency. My currency is kind of… my currency is like pictures. But the general currency which works for everybody is a formula. So if you were looking on a test, you'd hope for some kind a formula or you'd hope for a lot of words and a (gestures to the picture). But it would take a lot more words to make the picture proof communicatable.

So if proof has both a public and a private aspect, what is it that makes the proof prove? For Prof. B, it appears, at least in the case of the problem at hand, that the public and private aspects are connected through what I will call the *key idea* of the proof. As Prof. A says above, there is one and only one thing about the even function—its symmetry across the y axis—which implies that the derivative at x is the opposite at minus x. This key idea can be represented in different ways, one that resonates with his own personal thinking (my currency) and one that would be acceptable in a public forum (general currency). But there is one single idea which can be communicated in different ways. If we say enough words about the picture, we can render it in a form that will communicate to someone with appropriate mathematical background why the claim is true.

Not all professors generated a proof that the derivative of an even function is odd by communicating an idea. The next professor we consider simply sat down and wrote a proof using the definition of derivative (see Figure 3.2 below).

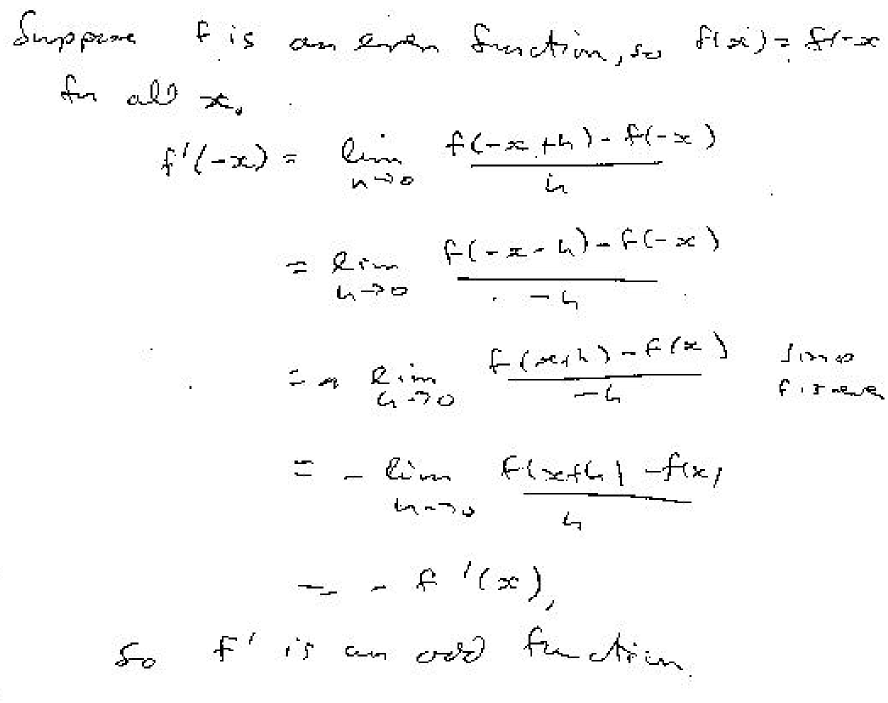


Figure 3.2: Prof. A's proof

However, even though this professor does not use a graphical argument to generate a proof, he knows that he *could* have. When asked if he had any pictures in mind when he generated that proof, he replied:

> Prof. A: No, I didn't do it geometrically. If I had had trouble writing up the analytic solution, then I would have drawn myself a picture. But I looked at this and I thought, if this is going to be true it's got to come out of the definition of derivative.

Prof. A's concept of proof, just like Prof. B's, is one in which the public and private aspects have an essential connection. He too recognizes that there is a key idea of a proof, even though the proof he produced was just cranked out mechanically. Later when he is looking at the graphical response he says:

> Prof. A: A mathematician at least could look at this and could write out a correct proof, because it is clear that there is an idea there.

Like Prof. B, Prof. A sees that there is an idea in the picture. He goes on to say:

> Prof. A: If I were going to use that picture, I would take it and turn it into a proof. Although if you do that, it comes down to pretty much this (#3).

So the idea embedded in the picture, when written out in precise mathematical language, looks like the argument that he gave using the definition of derivative.

Graduate students

The graduate students in the study were also able to produce a proof that the derivative of an even function is odd. All of the graduate students chose a formal approach, using either the chain rule or the definition of the derivative unless asked to consider alternative approaches.

As an example we take Grad. A, whose thinking was typical of the graduate students in the study. Like Prof. A, Grad. A was able to produce a correct proof using the definition of derivative. However unlike Prof. A, when he is asked if he had any picture in mind, Grad. A is not so certain he could come up with a graphical argument.

> I: And when you did this using the definition, did you have any picture in mind, or just symbols and definitions.
> Grad. A: No, it was purely symbolic on this level. I guess you could draw. You could take examples, you could do x squared I guess. Let's see. Any simple geometric reasons for as to why this happens. I'm trying to think… I can't think of an easy way to say it. There should be, though, but I am not the right person to talk to as far as geometry is concerned.
> I: Ok
> Grad. A: I mean maybe, if you look at a parabola, then it is sort of a flipped version… I guess as you take this limit then what is happening to your slope lines is that they too are becoming inverted. They are just getting flipped around in a sense and that makes them negative. And you could argue in that fashion.

Nonetheless (and we will see later that this differs from the undergraduate students) once Grad. A produces a graphical argument, he immediately likes it.

> Grad. A: That would be a perfectly fine argument, even for a graduate student or a mathematician. You can stop right there for me.

Like the professors above he sees how the idea embodied in the graphical argument is related to the formal proof he produced initially.

> Grad. A: Alright let's rigorize this, let's symbolically manipulate it because now we have convinced ourselves of the veracity of this claim. Let's go flesh it out and see how it works.

So like Prof. A above, even though Grad. A doesn't initially think of his proof as a translation of a idea, when pushed he not only can generate the idea but he also sees that the graphical representation of that idea is related to the limit definition representation, namely the latter is a fleshing out of the former.

While the graduate students as a whole are similar to the faculty, there is one graduate student who articulated her approach to the proof in a slightly, but ultimately fundamentally, different way than the faculty. The main difference between Grad. B below and Prof. A comes from the way they view the formalism they use in their proofs. The difference seems to be largely a matter of confidence, but the level of confidence indicates a subtle difference in how they view the activity of proving (at least in the context of the problem at hand).

Recall that when Prof. A chooses his strategy of using the definition of derivative, he is sure it is going to work. ("if this is going to be true it's got to come out of the definition of derivative.") Grad. B also decides to proceed formally (using the chain rule). However, unlike the professor, she seems less certain that her strategy will work.

> I: Can you say why you chose this way of (proving it)?
> Grad. B: Oh, that was just the first way that comes to mind. So, I mean, my general approach to proofs like this is—it says to prove something. It's got a bunch of words in it. I know what they mean. I always write down, "let whatever it is be whatever I'm supposed to start with." Say what that means, definitionally. And then see if I could see an obvious way to take that and get the definition of the next thing.

This graduate student, who is still taking courses and has not yet begun doing her own research, appears to view proving as following a procedure. She does not know, based on strictly mathematical grounds, that this procedure will yield a result. This view seems different from that of the professor who thinks the proof must come out of the definition. For the professor the definition embodies the mathematical idea that will lead to the proof, whereas for the graduate student, the mathematical formalism is a simply tool to connect the given to the conclusion.

Students

Unlike the teachers, none of the students were able to generate a complete proof. However, they were able to make some progress. Like the faculty, they began in one of two ways, either by starting with examples and trying to produce a formal proof, or by starting with a formal type of proof and getting stuck. However, unlike the faculty (and more like Grad. B) the students who tried to produce a formal proof did not do so with confidence that they could produce a proof. When they got stuck they resorted to pictures or examples but did not see how those are connected to the formal mathematical content they have been studying.

Consider first a student who starts with examples and can't produce the coin-of-the realm. He is asked how convinced he is that the claim is true based on the examples he generated.

> I: How convinced are you that it is true? What percent?
> Student C: What percent? I guess, I guess I know it **has** to be true. I mean, I don't know, I mean I guess I know its true, like100%. I know it has to be true. Because I've done derivatives so many times, you know. But I just couldn't really prove it. Because that's the way it is, but I don't know how to prove such a statement.

However, later when shown Response 1 containing different examples he says:

> Student C: This is just not convincing because you're not… you haven't proved anything. You're just showing examples. I mean how to do derivatives. You're not doing anything conceptual. I mean, examples are not proof. So it is not very convincing.

Like Prof. B above, Student C distinguishes between private and public senses of conviction. He knows, based on examples that the claim *ought* to be true, but is not able to produce the coin-of-the-realm argument. However, unlike the faculty member, the distinction between the public and private realms is not so clear cut. Below he is asked to explain the difference in his conviction based on the examples he generated and the examples in response #1.

> I: So let me ask you something, before when I asked if you believed it was true from your examples, you said you completely believe it is true, 100%. So how does that relate to what you said now?
> Student C: Well, um, I mean. I guess there is like this doubt in my mind. That it might not be true, but I mean since I know.. or I mean if that's how.. since I know. There's so many times, I guess… it has to be true…

> its true…I know its true. Its true. But, for I mean… we have the same problem because showing examples doesn't mean that its proved, right? You see what I'm saying. You're showing examples.
> I: But I'm just curious about the fact that on one hand you say you completely believe its true…
> Student C: Well I don't totally believe its true…
> I: Well, you said 100%, you can revise that if you want
> Student C: No, I'm not revising it. I still, I'm convinced it is true. But how do you go about proving that's true. It's harder. It's different.

The key difference between the student and the faculty member is that the student has not found the key idea of the proof. For the faculty the public and private realms are connected through this key idea. For the student, even though he has started the proof much like the faculty, there is nowhere else to go. Proving is "harder", and perhaps more significantly, it is "different". This appears to be an epistemological difference between students and the teachers above.

While some students started with hands-on examples, others began by trying to produce a formal-like proof. A difference between students and teachers, however, is that students tended to try to produce a formal proof without having an informal idea behind the formal or knowing they could produce one when pushed. For instance, Student L knows the definition of even and odd and has the idea to use the chain rule, but doesn't recognize that his calculation shows that the derivative is odd. He writes:

Define even function:

f(-x)=f(x)

Define odd function:

f(-x)=-f(x)

$\frac{d}{dx} f(-x) = f'(-x) \cdot -1$ by the chain rule

$= -f'(-x)$

At this point Student L has produced a correct proof, but he does not think he is done. He erroneously thinks he now needs to show that –f '(-x) is an even function. Unlike the faculty who either have the confidence that the result will come from the definition or else start with examples to get a sense of why the claim is true, this student is starting with formalism in the blind hope that he will get a result. When he gets stuck, he looks at an example of $f(x)=x^2$ which he identifies as an even function, and its derivative f'(x)=2x which he identifies as odd. This seems to convince him that the claim is true, as he writes "so works" on his paper. However he does not seem to connect this example to his written work (otherwise he would catch his mistake of thinking that f' should be odd, not even) nor does he seem to stop and think what any of his formalism means. He is so determined to write a formal looking argument that he doesn't stop to see whether he has one or not.

Why is it that students have difficulty generating a proof? The difficulty comes in part from how they conceptualize the underlying concepts of even, odd, and derivative. Students tended to see the graphical and formulaic representations of these concepts as related, but separate. Consider the following student who wrote down the wrong definition for odd (he had written -f(x)=f(x)). Below he explains how he was trying to remember the definition.

> I: So you drew the graph and then you didn't end up using it?
> Student F: I didn't totally use it. I tried to remember still. But yeah. I drew the graph to kind of see what I could kind of get out of it. I tried to remember the formulas, but without the graph, which is kind of dumb, but…

This student knows that the formula that characterizes an odd function is related to a graphical representation. He even draws that a graphical representation that could help him get the correct formula, but then he doesn't use it. This indicates that the student does not connect the graph with the formula, which he just retrieves, falsely, from memory.

Another student actually tries to use her graph of an odd function to recall the formula, but is still uncertain about whether her definition is correct. When asked why she responds:

> Student J: Because I wasn't sure if like, I was supposed to compare these two points around the origin. Like for this (even function) I was sure because that had to be that, because I just remember doing that. But for this one, I didn't know… I was pretty sure, but I didn't know which points I was supposed to exactly compare.

While Student F tries to recall the formula from memory, Student J is trying to recall a procedure (including making a picture) for remembering the formula. It seems that the graph is something she associates with a formal definition but not something that embodies meaning for her.

Consider in contrast the way Prof. B recalls the definition of even.

> Prof. B: Let's see, an even function. There is only one thing about it, and that is its graph is reflected across the axis.

The crucial difference between this professor's conceptual understanding and those of the students above is that for the professor there is one and only one important property of even function—its symmetry across the y axis. The graph and the formula are both representations of the same idea. And this idea provides grounds for a proof. It is clear here that the professor's privately held understanding of an even function is linked to what he believes would be a publicly acceptable proof that the derivative of an even function is odd.

Students' understanding of derivative is also a factor in their ability to generate a proof. For example, Student J above appears limited by the way he thinks about derivative. Consider the way he describes his work:

> I: How did you convince yourself that it was true?
> Student C: So my understanding of derivative is that you subtract the power by one. Right, so if you have an even function, the power is even, so it always comes out to be odd. That's my… my intuitive understanding of the problem. And then… I don't know… I tried to get somewhere, but I really couldn't, so I just write down the formula for the… I guess the definition for what the derivative is. So, that's what I have. And I couldn't go anywhere from there.

It appears from this response that Student C has the 'wrong' definition of derivative, based on his experience taking derivatives. But it is not that the student doesn't 'know' the definition of derivative. In trying to generate the proof, Student C wrote down the correct limit definition. The problem is that Student C does not see any way to use the formal definition to generate a proof. He ends up with two different kinds of approaches, one involving his polynomial examples, which makes intuitive sense to him, and one with the definition of derivative, which he thinks could lead to a publicly acceptable proof, but is aborted soon after its inception. The gap between the public and private arguments is too large for this student to bridge.

Summary

Thus far we have seen three different kinds of ideas, grounded in ways of conceptualizing the notions of even and odd and derivative, that could have the potential to lead to a proof.

The first kind of idea, which I will call a *heuristic idea*, is one which is grounded in experience. This is what most of the students have: looking at examples, using 'intuitive' understandings of concepts. This is also what Prof. B had when he initially thought about his proof. The heuristic idea is what gives one a sense that something ought to be true. Prof. B is not convinced on the basis of examples alone. Even Student C, who says at one point that he is 100% convinced based on examples knows that examples are not proof and says that the examples give him a sense that it "ought to be true". Heuristic ideas can lead to a private sense of conviction.

The second kind of idea, which I will a *procedural idea*, is one which when carried out correctly, like in the case of Grad. B, could lead to a proof but does not by itself give any sense of understanding. Start with a definition, fiddle around with it using algebra to see if you can get what you want to get. A procedural idea can lead to a proof that demonstrates that something is true, but may not give any insight into why it is true or even why it ought to be true. Procedural ideas can lead to a public sense of conviction.

The third kind of idea, which I will call a *key idea*, is the link between the private and public realms. This is the kind of idea that both resonates with experience and can be translated into the coin-of-the-realm. Key ideas both bring about understanding and can be rigorized appropriately. If one has a key idea, then one has both public and private senses of conviction and see how the two are linked.

Chapter 4: Data and analysis, part 2

The previous chapter focused largely on participants' behavior as they were generating a proof of the claim that the derivative of an even function is odd. There was a fairly big difference between students and faculty viewpoints, namely that faculty thought in terms of key ideas while students did not. This chapter examines how students, graduate students, and faculty evaluate five different approaches to proving the claim. Even though the burden of producing a proof is removed, students and teachers react quite differently to the different responses, based again on whether or not the participants have understood the key idea of the proof. This difference translates into differences in whether a response provides grounds for a proof, whether a particular argument shows 'why' the claim is true, and which kind of argument would be worth points on an exam vs. which would demonstrate understanding.

Proof evaluation

Response 1

Consider the following functions and their derivatives --

$f(x) = x$ odd	$f(x) = x^2$ even	$f(x) = x^3$ odd	$f(x) = x^4$ even
$f'(x) = 1$ even	$f'(x) = 2x$ odd	$f'(x) = 3x^2$ even	$f(x) = 4x^3$ odd
$f(x) = x^5$ odd	$f(x) = x^6$ even		
$f'(x) = 5x^4$ even	$f'(x) = 6x^5$ odd		

Note that for all even functions, the derivative is odd. We could continue for all powers (n = 7, 8, 9, ...), thus the claim is proved.

Faculty

Faculty like the examples in Response 1 to some extent because they are heuristic for a Taylor series kind of argument. Hence they see this response as more proof-like than the students who, as we will see below, simply say that examples are not proof. Consider the remarks of Prof. A:

> Prof. A: Well the first one is suggestive, but not convincing. It is certainly not a proof. It shows the student understands the statement of the question, but doesn't understand what the nature of the proof is. Nonetheless within that range, I think it is fairly close. They don't just take a finite number of examples, they… the student also generalizes and says this is true for every power. And that is getting closer to something that is a proof. You could get it for all polynomials, for example by… pretty readily from this. And indeed if you had Taylor Series you could get it for analytic functions.

Prof. A seems to be distinguishing between a heuristic idea and a key idea or a procedural idea. The idea embodied in the examples above is heuristic. The examples provide grounds to believe that the claim *ought* to be true, but does not demonstrate *that* or *why* it is true. However, note that Prof. A reads into the examples and sees them as grounds for a proof. This argument could work for all analytic functions (which are the kinds of functions students have seen). Prof. E makes an even stronger claim about how convincing the examples are:

> Prof. E: Response 1 is not convincing to me, but only for one reason. I understand that it could be a good idea to teach students in this way, but you can't assume all functions are polynomials. You have to put something else, exponent anyway.

For Prof. E the only thing that makes the claim unconvincing is that it is not general. He seems to think that showing examples would be a good pedagogical tool as long as one used some examples that are not polynomials. There is some indication that Prof. E here is referring to a private sense of conviction. Examples do not count as proof, but by looking at polynomial and non-polynomial examples, one can get a sense that the claim ought to be true, and that in turn makes the argument convincing.

Note that in making this comment Prof. E, like Prof. A above, reads more into the example than is there on paper. He makes a rather sophisticated move from the monomial examples to all polynomials. Monomials can be combined to generate polynomials which can approximate analytic functions. This behavior was typical of the faculty and several graduate students. They tended to apply their knowledge and read in between the lines of the proof responses in front of them, often attributing more knowledge to the writer of the proof than would be justified based solely on the written response.

Graduate students

The graduate students also recognize examples as heuristic.

> Grad. D: Response 1 and 2 I would give partial credit. Because they certainly know examples of odd and even functions and presumably because they cannot give a theoretical proof they tried to work out by example. So it is definitely a step in the direction of getting a proof. So the first thing that one does when one can't come up with a rigorous proof is to try to work out examples. And this particular response is in that direction. So I would probably give a couple points for that.

The fact that Grad. D sees the examples as "definitely in the direction" of a proof indicates that the examples could potentially link to a rigorous argument. Later she talks about how she would treat this problem in her calculus discussion section. She says that she would use examples to help establish that the claim is true even though on an exam she would expect an answer using formalisms.

> Grad. D: So, if it’s the discussion section, I would definitely first try a couple of examples before actually launching into the main proof. Because sometimes proofs are proofs. They are fine, except they are not very illuminating. And a better way of actually convincing yourself that a result is actually true is to work hands on an example. Which might not give you a clue as to what the most general rigorous proof might be, but at

> least it convinces the student that ok, at least what we want to prove is correct as we can see by examples. So if it is the discussion section, I would first start maybe with a little bit of this and then actually ask how to prove it in the general situation. In an exam of course that is not going to be… that's the place where you just write down what is correct.

Notice that at least in this case, Grad. D sees the role of the examples as not necessarily to generate a proof (they "might not give you a clue as to what the most general rigorous proof might be") but to give a sense that the claim is true. She seems to think examples are very important in discussion section, but draws a firm line when it comes to what she expects on an exam ("that's the place where you just write down what is correct.") Unlike Prof. A she seems less concerned with whether the examples are grounds for a proof (perhaps she does not see that they are), but rather sees them as a pedagogical tool for helping students develop conviction before "launching into" what will count as a "correct" proof in her course.

Students

While the teachers seem to think that examples are (or at least could potentially be) heuristic, that is to say provide grounds, for finding a proof, students are more suspicious of examples. Several students comment that while examples are good enough for high school, they are not accepted at the university level (which seems to ring true with Grad. D's comments above. Examples won't count for her on an exam). This distinction is important to students, many of whom admit that they are struggling with proofs in their college level calculus courses because it is more theoretical than their high school calculus course. Several students, like the one below who gives Response #1 zero points, want to make clear that they know that examples do not constitute a proof.

> Student L: I want your survey to indicate that we students who come from high school, we can see that a proof… that this is not really a proof.

The students are able to articulate reasons why examples are not proof, such as the fact that they are not sufficiently general. However, unlike the faculty, they fail to see that the examples are heuristic for generating a proof. Consider Student A below who also did not want to give any credit to response #1. Unlike Prof. A who says that the examples are close to a Taylor series argument for analytic functions, Student A does not see any argument that could be related to the examples.

> Student A: (reads #1 out loud) I'd give this zero. There's no, like, proof that if you continue on, like for these integers that you will see this pattern. It is just not really solid. These are just examples. There could be a counterexample along the line and you wouldn't have checked for it because you haven't gone through infinite number of integers. And you can't do that. So you haven't proved it for all n.
> I: Is it convincing at all to you?
> Student A: No. I just see it as an observation. That, oh, I can take the derivative. This is odd, this is even. It doesn't convince me. Plus (laugh) they don't even state what an odd function really is. They just said, ok, that's an even power that's an odd power so that's even odd. That is not convincing.

Here the student does not seem to have a private sense of conviction. The examples are just an observation. Unlike the way the teachers saw the examples, Student A does not see the examples as potential grounds for a proof. Without the concept of analytic functions or extensive experience with Taylor series, the examples before Student A are simply isolated examples, not the grounds for a more sophisticated argument.

Below is an example of a student, who unlike A, does have a private sense of conviction. However, she also does not seem to see that the examples are grounds for a proof.

> Student E: Ok. The first one, is pretty much what I was going for. It is pretty convincing, except it only shows examples. It doesn't really show why.

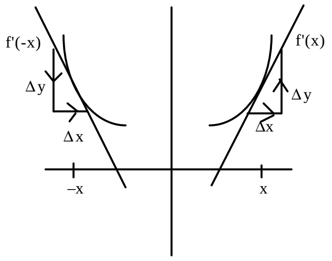
As we will see below, this was a common type of response for students. They tended to think that responses like #1 and 2 do not show why a claim is true while a response like #3 does. This contrasts with the faculty who think response 2 shows why the claim is true, and response 3 does not.

Another difference between the students and the faculty is that for most students, there is a difference between a response that engenders understanding and a response that would get a good score on an exam. Consider Student K below who had used examples herself to try to prove the claim was true.

> Student K: For this (1) I liked it personally because it helped me understand it. It is really clear. But I don't think the person who is grading it, the math teachers, will like it this way because it might not be true for all the odd functions.

The students' expectation of a formal-looking type of answer helps reinforce their disconnect between public and private types of arguments. Even those who generated examples to help make sense of the problem believe that examples would not get many points on a college exam, so they would avoid this kind of approach on an exam. The students do not seem to be aware that many teachers consider the examples, at least in this particular case, to be point-worthy because the examples could be heuristic for generating an actual proof.

Response 2



If f(x) is an even function it is symmetric over the y axis. So the slope at any point x is the opposite of the slope at -x. In other words f'(-x) = - f'(x), which means the derivative of the function is odd.

Faculty

In looking at teachers' comments about Response 2 we will see how important the key idea is to them. The picture is the proof, at least for this particular problem. Even if the argument is not rigorous, it is good because it captures the key idea.

> Prof. E: #2 I would simply accept, even if it is not rigorous. Somehow I think it is so useful that a student could think in these terms, I think, for his or her career, I think it is more important than their ability to write down these polynomials. So #2 I think I would simply accept without many questions.

The picture in Response 2 has mathematical value. The reason why teachers, on the whole, like Response 2 is that the picture conveys to them a sense of *why* the claim is true:

> I: What kind of response do you think would demonstrate the best understanding?
> Prof. A: Depends on understanding of what. Understanding of why even functions have odd derivatives, in some sense the picture does that best. In terms of understanding what constitutes a proof 3 and 4 are better at that and much less good at understanding why even functions have odd derivatives.

Even though Prof. A is not willing to call #2 a proof (several others teachers were), he says that the picture captures the 'idea' of the proof.

Not all faculty were not completely satisfied with response #2, but that was not because they didn't think that pictures could capture the idea of the proof. Rather the text that went alongside the picture did nothing more than describe, as opposed to *translate*, the picture. In the following complaint about #2, we see that the professor (1) thinks the picture is a proof (at least in the private sense), and (2) sees generating a publicly acceptable proof as a translation of the idea embodied in the picture.

> Prof. C: The picture is a perfect proof, but the paragraph below is not such a great proof. Why doesn't he translate the picture? He's saying listen, just trust me that this is true. (Starts reading outloud) f is even if it symmetric over the y axis. Fine. (reads second sentence) what is meant by opposite? reciprocal? negative? (Keeps reading) In other words, f prime of x is equal to minus f prime of minus x. Why? That is what you are trying to prove. He's repeating the conclusion. You're supposed to arrive at that conclusion by a logical process.
> […]
> That is the problem with most beginners. They don't know where to put their foot down, so to speak. They just say, you know, you know I see it, don't you see it. I say if you see it, explain it so that the other person can see it.

Note that the pr ofessor recognizes that the proof contains the idea. The author of Response 2 can presumably 'see' why the claim is true. The task then for the author is to explain that vision in terms that someone else can understand. Prof. C points out several places where misunderstandings could occur. What is meant by 'opposite'? What allows you to claim that f'(x) = -f'(-x)? However, it is clear that what the professor finds lacking is the careful reasoning to clarify and/or justify the idea conveyed in the picture, not the idea itself. Further he expects a proof to be a translation of the idea, even if the text that accompanies Response 2 does not do the job for him.

Graduate students

The graduate students, like the faculty, were able to see the key idea in Response #2. However, there was some disagreement on how to grade it. Some graduate students, like Grad. A below, thought it deserved full or nearly full credit:

> Grad. A: It (#2) is a fine argument and it can be rigorized very easily. In fact it is quite rigorous. It is in essence nothing more than this (points to his formal proof, like #3). Except it has intuition, so it should get full credit, I think. It has intuition and it's clear what is going to happen when you take. Ok, so maybe you might want to actually write the definition of the derivative down. So, I am not quite sure. Somewhere between 9 and 10 points. I would probably give it 10 if it is in a <first semester calculus> class I would give them full credit.

However, one graduate student who liked the graphical argument, was hesitant to give it full credit.

> Grad. B: I would like to give it full credit, but somehow I feel I'm just not allowed to. That a picture isn't good enough. It doesn't look like there is any math written down. I know, that is so stupid.

What is it that counts as mathematical for this graduate student? (Note this is the same graduate student who above thought that producing a proof involved following a procedure, starting from definitions and leading to the conclusion.) On the one hand, she understands why the claim is true and can see the picture demonstrates the key idea. On the other hand, she doesn't feel she is allowed to 'count' such a response because it is not mathematical. It appears that this graduate student is struggling herself with the question of what counts mathematically.

This struggle can be seen in some other comments she makes about the kind of understanding she would like her calculus students to have:

> Grad. B: I do, at some level, think it is important for them to learn the definitions of the derivative and stuff, I know that a lot of my kids have learned that but not understood why that has to do with it being a derivative. Like for me the limit definition really has to do with (gestures

> with her hand, using index finger to simulate a tangent line) taking the limit and getting a line at your point and I don't think they have associated those two pictures. I don't know why I seem to think that thinking of it as a slope is better than thinking of it as a limit, but I do at some level. I tend to think that you should draw a picture of what is going on to actually understand things.

We see here an example of a teacher struggling with how to teach the concept of derivative. She sees that the limit definition of derivative 'really has to do with' the tangent line. These representations of derivative are linked through a single idea. However she notices that students tend to not make the association. Further she thinks that the graphical representation of slope is somehow better because it engenders understanding, yet she does not feel she can give a picture argument full credit on an exam. This graduate student appears to be in transition between what, as we will see shortly, seems like a student-like perspective (pictures don't count as mathematical) and a faculty-like perspective (pictures, which embody the key idea, are significant mathematically.)

Students

Students tend to be more harsh about Response 2 than the teachers. They don't see it as convincing, and for several students, unlike Prof. A above, it doesn't give a sense for why the claim is true. First we consider two of the weaker students in the study. The weaker students care more about form. The picture proof doesn't look like a proof to them because it does not contain the kind of formal elements they expect to see in a proof.

> I: Are you… like if he… if someone asked you if this is true or not and gave this (2) as a reason, would you be convinced by it?
> Student C: I wouldn't be convinced. I mean, it doesn't look like there's enough facts… or enough calculation involved. I just think there should be more like, at least given like… he could prove… I mean maybe… I don't know… use the definition of derivative and derive from there, then I

> would say it is a full credit proof. But… right now from this response he is just saying it as it is. He isn't saying, "why" it is the way it is.

Further, the picture proof doesn't even seem to give a sense of why the claim is true, which is opposite from what the teachers said.

> Student E: Um, ok, for this one, um I don't really think it is very convincing. It doesn't really state anything. It just has a picture. It just basically tells what the picture does. And again it doesn't explain why. It doesn't answer the question.

It is hard to tell why the students above don't see that response #2 is convincing or shows why the claim is true. It could be that they simply do not have the key idea so they cannot recognize that the picture conveys that key idea. However, the distrust of pictures was also held by students who did seem to follow the picture proof. Consider the remarks of the student below.

> Student L: It is not a full proof, but it is a proof, it is graphical proof. And sometimes, I don't know if he ever weasels out of a proof by doing a graphical proof. I guess for hard ones. But usually he gives it as part of the proof, or maybe as something to explain. It helps to have something like this to actually know that it is right. Or to give the idea that it is right. That this isn't something out of nowhere. Sometimes people have an easier time understanding graphs than formulas. And then to go into the formulas. I think you need to incorporate the definition of derivative to show it. I don't think this proof is enough but it would get some points.

Student Student L seems torn about whether to call the picture proof a proof. "It is not a full proof", but it is a "proof", yet pictures seem to be a way of "weaseling out" of doing a proof. This student does seem to recognize that there is an idea in #2, however the fact that there is an idea doesn't seem to be as important to him as it is for the faculty. For the student the picture gives some sense that the "real" proof isn't coming out of nowhere. This student seems to stop short of saying that a picture proves, as the faculty above say.

Response 3

Want to show if f(x) = f(-x) then - f'(x) = f'(-x).

f'(-x) = $\lim_{h\to 0} \frac{f(-x+h)-f(-x)}{h}$ by the definition of derivative

= $\lim_{h\to 0} \frac{f(x-h)-f(x)}{h}$ since f is even

Let t=–h

= $\lim_{t\to 0} \frac{f(x+t)-f(x)}{-t}$

= – $\lim_{t\to 0} \frac{f(x+t)-f(x)}{t}$

= –f'(x) as desired.

Faculty

All the faculty thought that response 3 was a correct proof and thus should get full credit on an exam, however some of the faculty did not like response 3 as much as some of the other responses.

> I: And what did you think about response #3?
> Prof. B: (laugh) Wildly too…Way too complicated, but correct in every detail. You get points for correctness even if it is not the shortest proof.

The proof in Response 3 is "way too complicated" for expressing the simple idea that the slope at x is the opposite of the slope at negative x. It appears that for Prof. A a good proof is one that elucidates the key idea rather than obscuring it.

Some of the faculty assumed Response 3 was correct without checking the details. The form of the response, with the limits and algebra, looked right.

> Prof. A: Unless there is something hidden in it, then I think this is fine. I think it is10 points. I tend to assume that if something has the right form and the right elements in there, then the details must be correct.

and

> Prof. C: I don't have my reading glasses, but that is a proof to the extent that I can read it.

So the fact, as we will see below, that students judge a proof by its form does not distinguish them from the faculty. The main difference is that faculty, when pushed, are able to check or generate the details (for instance Prof. A above had generated a correct proof like Response 3 on his own).

Graduate students

The graduate students also thought that response 3 should get full credit, but in discussing the merits of the proof, several graduate students commented something to the effect that a combination of responses 2 and 3 would make the proof rigorous.

> Grad. D: Well, a combination of response 2 and 3 would make it rigorous I think. Here they have basically just drawn a picture and essentially measured the length to see that the difference quotients are basically the same or one is the negative of the other, actually. So then in the limit they should also be the same. So that is basically what is going on. However, I just need them to realize that they can't apply the tangent of the slope interpretation of derivative for every problem that they run into. And that there is a theoretical definition out there which is specifically meant to work in every possible situation and this is one situation where they need to use it.

> Grad. A: And if someone did this proof (#3) they would get full credit too. I might make a remark, like you could draw a picture too. So the true proof would be this and a picture. They would get extra brownie points or something.

Why is it that the graduate students seem to think a combination of 2 and 3 is rigorous, or a 'true proof'? Again, it seems the link between the formal proof and the picture or idea is important in constituting a proof. Student K was asked, by email, what he meant by a 'true proof'. He responded:

> Grad. A: […] there are two parts to understanding. The proof is the means by which one logically arrives at a conclusion and argues the veracity of the claim to others and oneself. But there is intuition as well.
>
> In testing a student, it would be best to have both. The "picture" does not make the proof more "true." It only demonstrates that the student has the feeling, the right picture in mind. I prefer the answer with a picture and a proof because it shows that the student has both necessary aspects in mind. I did not mean "true proof" in a literal sense.

Grad. A seems to be struggling with what he means by proof. On the one hand, he calls the combination of the picture and the algebraic proof a "true proof" and says that both are 'necessary' aspects of proving. On the other hand, he says he doesn't mean 'true proof' in a literal sense, reserving the word "proof" for a logical argument. We see in Grad A's comments that he recognizes both a private and a public aspect of proof—with a proof, one argues "the veracity to others and oneself". But we see also that there could be something more to a proof, the intuition or key idea, which links the two together.

Students

Most students find #3 more convincing than #2 and think #3 is what teachers want on an exam (which is not true, for instance, of Grad. A). This is true both of students who don't see the connections between the two and those who do. Consider first a student with a fairly weak understanding.

> I: What makes this (#3) more convincing than this (#2).
> Student E: Because it actually shows why. It basically, this one just has a picture, and it doesn't really give any basis for the proof. This one (#3) shows exactly why the definition is correct. I guess you could have added the two. Like, um, put the picture and then you know use the definition of the derivative and then stated that at the very end, combined them. But just the picture alone doesn't say much about the answer.

While the student talks about combining responses #2 and #3, it seems that the picture for her has only heuristic value. It does not help show why the claim is true. One interpretation of Student E's response is that she does not understand response #2 well enough to recognize that it shows why. In fact, later in the interview, she is asked to explain the picture, and she has difficulty doing so. The excerpt above, then, seems to reflect that the student, not having developed her own private understanding of the proof, relies on relatively superficial publicly sanctioned criteria for a mathematical argument.

However even students with a strong understanding thought more highly of Response 3 than the teachers. This was the response they were trying to produce, even if they did not do successfully. It had all the properties of what, to them, was a perfect proof.

> Student D: (about Response 3) Clear as a teardrop. Top of the line. 10 points. Both convincing and understandable.

Below we consider a student who at first sees Responses 2 and 3 as separate, and then comes to see their connection. Even though he appears to see that Response 3 is a more formal version than Response 2, as several teachers above commented, this student sees Response 3 as being more convincing, and ultimately more mathematical, than Response 2.

> I: So which is more convincing so far of
> Student A: (pointing to #2 and 3) this one and this one?
> I: Yeah.
> Student A: Um... I still think this one (#3) is more convincing because it also states what a... well, kind of like states what a derivative is. And this person (#2) didn't really show what a… or didn't state what a derivative is, but they kind of showed it. Oh wow...
> I: Why do you say wow?
> Student A: I'm just seeing, I guess, the two different tracks you can take... the two different roads you can when teaching this. I understand your studies I guess (laugh).
> I: Can you say more about that?
> Student A: Well, I mean... its just that this (#3) seems more anal. And this (#2) seems a little bit more relaxed. But they both kind of show the same thing. (point to #3) More difficult. (point to #2) less difficult. Um, um,

> hmm... I think if you want to go into math, I mean if you are a math major, both of them are necess… both of them.... I think you should know both of them. But this (#3) more so than this (#2).

The reason that students like #3 is that it looks like a proof. Students think the faculty like formal looking language, such as 'if' and 'then' and 'therefore' (Student E). Moreover they believe that thinking mathematically entails formal manipulations.

> Student A: Let's see... think mathematically... yeah, that's one thing I thought I didn't have when I took <second semester calculus>. I couldn't really think in mathematical terms. I guess thinking mathematically is seeing... like... oh...being able to manipulate, if you have some function, you can turn it into another function using all sorts of little tricks. If you are thinking mathematically then you have all these tricks in your head then you could pull it from your pool and utilize it. Yeah.

Consider in contrast the remarks of the following graduate student. He sees the formal aspect of mathematics as "details", which while important, do not really capture what mathematics is about.

> Grad. A: But again, later on, (after calculus) the more math you do the more you realize your intuition can go wrong. So you need to have somehow this checking point. This rigor that needs to check your intuition. But on the other hand you shouldn't let it go so far that all you care about are details, because then you're not doing math.

This comment echoes the comments of Prof. A above about technically correct arguments in which central ideas get lost in a welter of details.

The fact that students tended to judge a proof by superficial aspects, such as whether it had the right language or amount of formalism, does not distinguish them from their teachers. One famous professor 'fell' for the false-formal proof because he thought it looked right. And several teachers accepted #3 as a proof without checking the details. The factors that seem to distinguish the students from the teachers were that (1) the faculty, when pushed, were able to move past the superficial level, and (2) students tended not to see that their private ideas about why the claim was true could be connected to a publicly accepted proof.

Response 4

Given f(x) is even, so f(x) = f(-x). Take the derivative of both sides. f'(x) = -f'(-x) by the chain rule. So f'(x) is odd.

Faculty and Graduate students

All the teachers considered Response 4 a proof, even if they didn't generate it on their own.

> Prof. B: Response 4 shows the clear thinker. It is obvious to this person what to do and why it is true. I'm back here (response 2) showing why its true because the picture is true. But if I had more time this (4) would be the way to do it, short and sweet.

Some graduate students have mixed feelings about Response 4. On one hand they think it is a proof, and a nice slick proof at that. However they don't think it is explanatory or pedagogical.

> Grad. B: I think it (4) says more about how proofs are done than understanding the problem in particular. Because it has sort of been drilled into me that if you don't know what is going on, you should (switches tone of voice to sort of mimicking tone) write out the definitions of everything that has been stated in a problem, look at the definitions, see if together they add up to the definition that you need.

> Grad. A: Ah, yes, this (#4) is what you should do. This is what I was trying to do, goddammit. Alright, this would be a very nice response, cute. This (#3) is a little better because it leads more to the picture which explains more. Response 4 is even more algebraic. It is even more anti-intuitive in a certain way. But it is just as valid. It is just not what I would want to do.

These quotes seem to indicate that what the graduate students value in the proof, at least for the purposes of teaching collegiate calculus, is the key idea, which is more transparent in response #2 than it is in response #4. However several teachers, such as Prof. P above, said that on a test they would be looking for something like #4 because it is easier to grade.

Students

Unlike the teachers, many students in the study seemed to have difficulty following the argument in Response 4. Without a sense of understanding, they judged whether Response 4 looked like a proof based on superficial aspects, such as its length.

> Student B: Ok, this is probably not as good because… I think it's a little too short. Let's see. They show they know what an even function is, but they are just taking the derivative of both sides, and they don't really show how they do it, they just put a little… well, I don't know… I think if I had done this, I would have defined an even and an odd function first. They are just like, oh, so f prime of x is odd. And they don't really elaborate on it that much.

The next student also had difficulty following Response 4. Interestingly, she had responses 2 and 3 and seemed to understand them, saying that those responses convinced her that the derivative of an even function was odd. However, in her struggle to understand response 4, it appeared that she began to doubt the veracity of the claim. She reverts to examples to check whether or not even functions have the property that f'(x) = -f'(x). Below she was asked about the difference between her reading of Responses 2 and 4.

> Student K: I get this (2) right away, but I couldn't get this (4). I didn't think this was right at first.
> I: But that was just interesting to me because here (2) you thought it (the claim f'(x)=-f'(-x)) was right, but then five minutes later you
> Student K: (laughing)
> I: seemed to not think it was right.
> Student K: Yeah, yeah. I didn't see the connection at all. But I got this (2) right away. f prime of negative x because the negative x comes from here (pointing to the left side of the x axis.) Then the derivative of f prime x (tracing her hand along the tangent line) comes from here. It makes sense to me. But it doesn't make sense then (when reading #4). That's interesting for me too (laugh).

It seems this student, who happens to be a fairly strong student in her calculus class, doesn't connect the picture idea to the formalism in response 4. Even though the picture seemed convincing earlier, she doesn't think in terms like the faculty that there is only one thing about the even function which is important here, and that the chain rule is just a translation of that basic idea.

One possible explanation for why students had difficulty following response 4 is that the chain rule step is not explicitly stated. Because of this several students were shown a more elaborated version of response 4 in addition to the version listed above:

> Let g(x) = f(-x). By the chain rule g'(x) = f'(-x) (-1) = - f'(-x). f(x) is even, so f(x) = f(-x). By substitution f(x) = g(x). We want to show f'(x) is odd, i.e. f'(x) = -f'(-x). Differentiating both sides, we get f'(x) = g'(x) =f'(-x). So if f(x) = f(-x), then f'(x) = -f'(-x), so f'(x) is odd.

However, none of the students thought this response was better than the original Response 4, and one student said it was more confusing.

Finally we consider Student A who, like Student D, said he did not like Response 4 because it was too short. Student A was interviewed again, about a year later, and he reported that he was asked to prove a similar claim by a friend. To his surprise, he generated a proof just like response 4.

> Student A: It's actually kind of funny. I was trying to help a friend out, in <first semester calculus> a couple of months ago and one of the questions was this exact one. I think it was the derivative of an odd function is even. Something like that. So then the way I answered it was like response #4. And I remember when I first saw this, for your study, I didn't like it, I think (laugh) so it's kind of ironic how I just kind of switched like that.
> […]
> I: Do you have any idea what changed?
> Student A: I think it was just the realization that you could just take.. it's just straightforward. You can take the derivative of both sides, then its rather obvious that the derivative of an even function is an odd function.
> I: Do you remember why you didn't like it?

> Student A: I think it was too short (laugh). I think I was biased by #3 because I was so used to proofs. And I mean, I thought this was how professors wanted it, #3, and any deviation would get lower points. But apparently not.

The interesting thing here is that even though the student produces a chain rule argument, he still thinks that a proof should look something like Response 3. By his own admission, Student A says he has been heavily influenced by a particular view of how mathematics should be presented.

Response 5

f is even, so f(x) = f (-x).
Multiply both sides by -1
- f(x) = - f(-x)
Factor in -1
f is even so we can substitute f(-x) = f(x)
f(-x) = - f(x)
Take the derivative of both sides
f'(-x) = - f'(x)

Faculty and Graduate Students

Most of the teachers gave response 5 zero points because it was wrong. For instance Grad. C found two steps that were 'wrong', line 4 which says to 'factor in', and line 8 which takes the derivative. He says he would simply write 'no' next to those lines and give zero points. One faculty member was a bit more generous with points, but still he thought the proof was fundamentally wrong because there was no "correct proof" that #5 was close to:

> Prof. A: I would give this about a 2. And this would be an example where the student would be very unhappy about the grade. And they would first of all would have some trouble understanding why you couldn't do that. And that could be explained, and they would understand that, but then they would say "that's only… that's an arithmetic mistake so you can't take 8 points out of 10 for an arithmetic mistake." Um… and you would respond, "but there is no correct proof that this is close to." It's not like they were doing a correct idea and they just made a little blunder in it. I don't think.

Notice that again, it is the presence or absence of an idea which is important to the professor ("it is not like they were doing a correct idea"). Notice also that as he reads response 5 he is thinking about whether or not there is a proof the argument is "close to". Just as he read into Response 1 to see that there was a proof it was close to, he reads into Response 5 and finds no such proof.

As noted earlier, one faculty member, Prof. E, 'fell' for #5, apparently looking only at the form, but when pushed immediately saw the mistakes and gave the response zero points.

Students

Most of the students in the study found Response 5 suspicious, if not wrong. However, even so, several students gave response 5 higher points than the teachers did.

> Prof. C: I would give him a 7 because first of all why did he multiply both sides by negative 1? [...] Actually professors would give more points than me because they proba bly understand it deeper than I do.

It appears this student, not confident of his own understanding, believes that this formal looking response could contain a valid argument.

Even the students who are suspicious of the argument are suspicious, not because the argument doesn't contain an idea like the teachers thought, but because the algebra is wrong.

> Student E: Ok. For response 5, I'm just gonna go… huh? Um… It seems like all they did was just you know put algebra and try to figure out something. But when they said substitute this f of x equals that, it doesn't seem right. Because they put f of negative x equals f of x, but they have the same thing for both sides, so it would just be like f of x equals you know f of negative f of x, and that's like not what they wanted. So, I don't think that's a very good response at all. I mean they show their work, but

> it doesn't seem right. They got to the right answer, but it seems kind of fishy. I don't know, the professor wouldn't give very many points, probably like 3, 2 or 3 points. I would probably give it like 1 or 2 points.
>
> I: Why do you think the professor would give more than you?
> C: I don't know. He seems kind of lenient with points. Like ok, if you put kind of key words, then he is like oh well. He likes sentences, I don't know why.

Note that this student thinks the response will get a few more points than she would give on the grounds that it has formal looking elements. In fact, her professor (Prof. A) did say he would give two points for this response.

As we saw with Response 4, the fact that Response 5 looked suspicious to some students made them doubt the veracity of the claim. For instance Student KY, who said the claim was true based on the arguments in Responses 2 and 3 begins to doubt the claim after reading Response 5. Below he explains why.

> I: But then when you looked at, when you were considering this one (#5) somehow you thought this (the claim) was wrong.
> Student A: (laugh) I just... I don't know. I did, didn't I? Hmm... I guess this (#3) is more impressive looking, maybe. Maybe it just... This (#5) looks kind of minisculish, while this (#3) looks at a higher level so then I almost automatically accept this. I guess that is more of a psychological reason. Just because it looks more impressive. You have limits in there, the definition of a derivative.

Like Student K, Student A laughs at himself for seemingly contradictory behavior. when reading Response 3, he thought the claim was true; when reading Response 5, he was not sure. The fact that he turns to concrete examples indicates that he does not yet have an understanding of why the claim is true. Thus he resorts to more superficial aspects of the arguments, such as the type of language that they use, to judge if the argument is a proof or not.

Summary

The faculty, who have the key idea, see that the informal types of responses (responses 1 and 2) are good insofar as they can lead to a proof. Response 1 can be seen as a heuristic argument that can easily be made to work for all polynomials, and thus for generating a Taylor series argument that holds for all analytic functions. Response 2 actually captures the key idea of one possible proof. The thing that sets Responses 3 and 4 apart from responses 1 and 2 is that they are written in a language that is acceptable in public settings. Response 4 is slicker and thus more elegant than Response 3, which they see as a direct translation of the key idea expressed in Response 2. Response 5, which doesn't contain a correct mathematical idea, is simply wrong.

The graduate students respond much like the faculty, except some of them struggle with the question of what kind of response should be encouraged on an exam. Even though Grad. B liked response 2, she didn't think she should give it full credit because it didn't look mathematical. Even though Grad. A thought that Response 4 was a slick argument, he thought Response 3 was better insofar as it was a direct translation of Response 2.

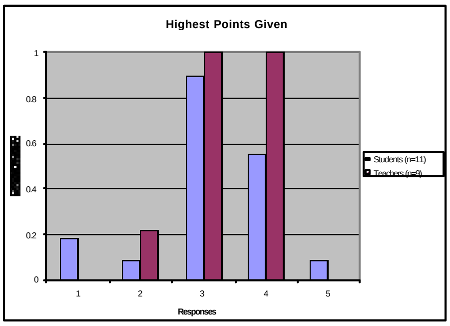
The students are much harder to characterize as a group since they varied in their level of understanding. Still several themes appear. First, students seem to think that Response 3, the most formal looking response, is convincing because it shows why the claim is true (while teachers thought that Response 2, the one that has the key idea shows why). The students do not seem to distinguish between public and private senses of conviction. Either the response is convincing or not, and the more formal the response, the more convincing it is.

Second, the students, who think the formal type responses are more proof-like than the informal responses, think (correctly) that their teachers want to see formal type responses on an exam. However, they underestimate the value their teachers will place on the informal responses because they do not see them as heuristic for generating a proof.

Third, students do not seem to really be convinced by any of the arguments. As we saw with Student A and Student K, as soon as they see an argument that they don't understand (like Responses 4 or 5), they try examples to see whether or not it is really true that the derivative of an even function is odd. That seems to indicate that the conviction they reported when reading, for example Response 3, was not deeply held.

All of these observations seem to point to the fact that the students, who do not have a deep understanding of the mathematics involved in this proof problem, attribute things to a formal looking response that the teachers do not. The formalism takes on a life of its own, lending a sense of conviction, demonstrating why a claim is true, and looking like the kind of response that their teachers will want to see on an exam. In short, what is mathematical to the students is the details that the professors see behind, and ultimately value less than the ideas that the details carry.

This difference between students and teachers can be seen clearly in the graphs below (Figure 4.1). These graphs contrast teacher and student evaluations of the 5 different proof responses based on (1) which would get full credit on an exam, and (2) which would show best understanding.



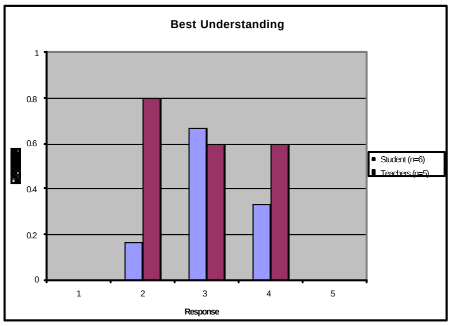


Figure 4.1

We see that for students (light colored bars) there is a correlation between what they would give highest marks as a good proof and between what they take to show the best understanding (the shape of the bars is the same on both graphs). In contrast, the views of the professors (dark colored bars) is almost the opposite. What they take to show the best understanding is the opposite of what they would give the highest points.

The picture that emerges from this data is something like this. Privately, that is to say, outside of the mathematics classroom, mathematicians value ideas. When constructing a proof or reading a proof, it is the key idea that they most value because that gives a sense of understanding, a sense of why the claim is true. However, the message that students get, in part from what appears to be expected on an exam, is that the formal aspect of mathematics is what is really valued. The students, who may not have the mathematical sophistication to see the ideas behind the formalism begin to attribute things to the formalism that the teachers do not. In particular, they believe that the "rigamorole" (in the words of Thurston) is mathematics, not just the language for expressing mathematical ideas.

Chapter 5: Implications for teaching

The goal of this dissertation is to compare the views of proof held by entering college students and their two types of teachers, graduate student teaching assistants and mathematics faculty. I have done so by analyzing the way students and teachers (1) attempt to prove the claim that the derivative of an even function is odd and (2) how they evaluate different proofs (and proof-like responses) that were provided of that claim.

The data indicate that in some ways teachers and students behave similarly. Both groups distinguish between private and public arguments. If people do not know how to produce a proof directly, they try informal approaches, such as trying examples or looking at a graph. Both groups recognize the goal is to come up with an argument that is sufficiently rigorous for the particular mathematical setting they are in (in this case a university level calculus course).

The difference between the groups, and I believe it is a significant difference, comes from their ability to see connections between the public and private realms, that is their ability to recognize the key idea which both gives a sense of understanding and can be made into a suitably rigorous argument. The professors in the study think in terms of key ideas, or at least know that they can, while the students do not.

The key idea in the case of this problem is that the symmetry of the even function implies that the slope at x will be the opposite of the slope at minus x. The picture (Response 2) captures this idea and to some of the faculty (e.g. Prof. C) counts as a "perfect proof." Response 3, which at least one faculty member thinks is too tedious, is a proof because it is a translation of the key idea into the coin-of-the-realm. Response 4, while considered a good and slick response by the teachers, was not considered by all teachers to be a good response for students: "This (#3) is a little better because it leads more to the picture which explains more." (Grad. A)

Even the approach of looking at examples, Response 1, is in the direction of a proof, although in this case it does not lead to a sufficiently general argument. Of course it takes a considerable amount of mathematical sophistication to see that Response 1 could lead to a proof, but the point is that one of the significant criteria that the faculty were looking for was whether an argument could lead to a proof (in the words of Prof. A, is there a proof that the given argument is "close to"?). Response 1 has the potential of being refinable to a proof, so it deserves some points. In contrast, Response 5, the false formal-looking proof, does not have that potential, so it should receive no points. To the teachers, the formalism counts only insofar as it conveys mathematical ideas.

In contrast, the students think that formalism, the public face of mathematics, is what counts mathematically. Students are not always able to come up with a formal argument, and they try informal approaches like looking at examples. However, unlike the faculty who see informal arguments as grounds (or at least potentially grounds) for a formal argument, the students seem to not see this connection. In the words of Student C, when faced with examples that he finds 100% convincing, "How do you go about proving it is true. It is harder, it is different." Some students, who do not seem to recognize the key idea in the case of what is to them a difficult calculus problem, do not recognize a picture proof as mathematical and think the chain rule argument is too short. Although judging a proof by superficial aspects does not distinguish students from faculty (e.g. Prof. C and Prof. A at least initially thought Response 3 was a proof because it looked right), the fact that the students seemed to equate the formalism with understanding stands in contrast to the faculty who say that the kind of answer that showed the best understanding is different from the kind of answer that would get the most points on an exam (see Figure 4.1).

These differences suggest that there is a communication gap between the faculty and students in terms of what makes a proof a proof. To the faculty, the most important characteristic of a proof is the key idea it contains. To the students, the most important aspects are things like language ("therefore", "thus"), symbolism, and length. Even a graduate student who is able to identify that a graphical argument demonstrates why the claim is true is reluctant to give it full credit because "it doesn't look like there is enough math written down. I know that is stupid." (Grad. B) Unless the situation is remedied, there is a risk that students will complete their mathematical education without seeing that mathematics is about ideas.

The focus of the remainder of this chapter is on the practical question of how to communicate this view of mathematics. Drawing on data from the study, I address the question of why the problem is so difficult and sketch some possible approaches.

What is the connection between a professor's view of proof and the message he or she sends to his or her students about the nature of proof in the classroom? As one might imagine, the connection is anything but straightforward. To illustrate this, we consider the case of Prof. B, who said he thought pictures were important and, as class visits bore out, used them liberally in his teaching. As we saw earlier, Prof. B, claims that pictures prove:

Prof. B: Pictures can prove things, that's what convinces you.

Moreover, to Prof. B pictures are more fundamental, mathematically speaking, than formulas:

> Prof. B: [The picture] is more fundamental. This is my opinion. It is more fundamental than this (pointing to formula).

In describing his own teaching, Prof. B. emphasizes the role of pictures. He says students are taught (presumably in high school) not to believe in pictures and he wants to "unteach" them.

> Prof. B: But students take a while before they believe in the picture, strangely enough.
> I: Why do you think that is?
> Prof. B: Well, because they're taught not to. That's part of it. I'm unteaching them, the best I can.

However, with regard to his testing practices, Prof. B thinks differently. Even though he teaches using pictures and thinks pictures are "more fundamental" than formulas, when it comes to the kind of answer he expects on an exam, Prof. B prefers formulas. Why? Because they are easier to grade.

> I: Which do you prefer?
> Prof. B: On an exam, this one (4). In the picture thing, you have to look at the arrows (why is it going down, because it is negative) […] if you read this you have to look at it, get into a person's world.

I do not think this is an example of a professor being lazy. I think Prof. B's comments raise an important question: what does it take to get inside a person's private world and to be able to evaluate the validity of their private arguments?

The validity of picture-based arguments was questioned by other professors as well. For instance Prof. A, who thought that Response 2 demonstrated the best understanding of why the claim was true, also mentioned that he would have been happier with a picture that had secant lines instead of tangent lines.

> Prof. A: It would actually have been, I think, slightly better if instead of drawing the tangent they would have drawn the secant line and looked at delta x and delta y because the fact that delta x has the same sign in the two cases whereas delta y has the opposite sign is clearly true for the secant line and it is not quite true, not quite clear why it is true for the tangent.

Passing to the limit is a nontrivial step, and it might not be clear from the picture alone that the student knows this.

Prof. C also comments about how difficult it is to tell how much a student understands from what he or she writes. Even though he said that the picture was a "perfect proof," he wasn't sure that if a student produced a picture like the one in Response 2 that he or she would be certain to really have a good understanding of the proof.

> Prof. C: Since he could draw the picture, he probably understood more than he could write down. Give the student the huge benefit of the doubt – that he knows the derivative is the limit of the difference quotient, and this difference quotient is clearly the negative of this difference quotient because this is the opposite of the other— which would be at least half [of the points], maybe a little more. But if you take a hard nosed attitude, you might give a third [of the points]. A lot of people could draw the picture as just a knee-jerk reaction. Draw it without really knowing why.

So what does this mean in terms of recommendations for how we should teach or examine students about proof? Even if it is the case that mathematicians value key ideas in their own work, it may not be so clear what it would mean to teach key ideas in the classroom. Below we consider a few possible approaches.

Two professors, Prof. D and Prof. E, addressed this issue by saying they would prefer to give this kind of question on an oral exam rather than a written one. They, like Prof. B, said that with a written exam it is difficult to get inside the students' head to see how much they understand. The benefit of an oral exam is that one can push students to see if the relevant connections are really there. Consider first the remarks of Prof. E who comes from Russia.

> Prof. E: I think it is more common [in Russia] to give oral exams in mathematics. I think it is a good idea because it is very unclear how to give credits, for example, to answers, for example to this question, (unintell) on a written exam. Even #4, which is my favorite, I don't know whether he or she simply remembers how it says in the textbook, or whether she knows, for example the general (unintell). So oral exams I think are much I think much better because you can communicate with the student. Get a much better idea of what he or she understands and what he or she doesn't.

Prof. E, by the way, was the one who said that if a student gave a picture like Response 2, even in an oral exam setting, he would accept it without question because "it is so useful that a student could think in these terms" if he or she is to pursue mathematics. Unlike Prof. C, he is convinced from the fact that the student generated the picture that the student has a good understanding of the proof idea.

Prof. D also advocated oral exams for testing a student's understanding.

> Prof. D: You can also ask on exams, but then you have no way of checking if it is just memorization or understanding. But if you have them in front of you at the blackboard then you can actually ask the question, you can pick the proof apart, you can ask 'why do you go from here to here?' And if they learn by heart then they fall apart.

Prof. D had taught calculus many times and actually used oral exams with his large (over 200 students) lecture courses. His practice can serve as an example of how oral exams could function in such a course. Prof. D's approach was to cancel lectures and discussion sections for a week and employ his teaching assistants to help him administer the oral exams. Students were given a list of problems they were supposed to answer in the oral exam. At the exam itself, they and their examiners shared the responsibility of choosing which questions they would answer in the limited time. The students had the option of picking another question than their administrator at the cost of some percentage of points.

Even if one is not prepared to change one's testing practices dramatically, one can still attempt to ask questions in a way that could help facilitate students' development of an appreciation for key ideas. The task central to this study—prove the derivative of an even function is odd—is found in many standard calculus books. For instance in the text used in the courses of the students in this study (Stewart, 1998), it is given in a section on the definition of derivative, implying that the author (as Prof. B suggested in his comments) would prefer students to answer with a response like Response 3.

In contrast, in Hughes-Hallett's *Calculus*, we find the question worded somewhat differently: "Looking at the graph, explain why if f(x) is an even function then f'(x) is odd." (Hughes-Hallet, 1998, p. 112) The task has been changed from proving to explaining why, and from the wording it appears the author wants students to produce a response along the lines of Response 2.

In light of the study reported here it seems that in order to help students develop the key idea of this proof, and a mature view of proof in general, it would be preferable to ask the question in a way that would get students to generate several different kinds of responses and to understand the connections between them. One could ask, for example:

> Is the derivative of an even function odd? Give at least two arguments to support your answer.

Ideally, this question could be asked in a discussion section so students could, after generating their own responses, compare those responses with others. (If they do not generate all the responses used in this study, they could be given them afterwards.) The discussion of these arguments, moderated by the teacher, could then focus on how the arguments are related or, assuming students will have difficulty generating a formal argument themselves, on how to render students' informal understandings into arguments that would be acceptable in a public forum. The following questions could be included in such a discussion:

- How would you write your informal understandings in a way that would be convincing?
- In your own words, what do the formal arguments (limit definition, chain rule), convey?
- How are the informal and formal arguments related?

As this study suggests, students do not seem to share the same view as mathematics faculty about what counts mathematically. Highlighting the link between informal and formal arguments might help address this problem, allowing students to develop a more mathematically correct view of proof and a deeper understanding of mathematics, in general.

Bibliography

Alibert, D., & Thomas, M. (1991). Research on Mathematical Proof. In D. Tall (Ed.), *Advanced Mathematical Thinking* (pp. 215-230). Dordrecht: Kluwer.

Balacheff, N. (1988). Aspects of proof in pupil's practice of school mathematics. In D. Pimm (Ed.), *Mathematics, Teachers and Children*. Kent: The Open University.

Boyer, C. (1989). *A History of Mathematics*. New York: John Wiley and Sons.

Brown, J. R. (1997). Proofs and Pictures. *British Journal of Philosophy of Sciences*, 48, 161-180.

Chazan, D. (1993). High School Geometry Students' Justifications for Their Views of Empirical Evidence and Mathematical Proof. *Educational Studies in Mathematics*, 24, 359-387.

Davis, P., & Hersh, R. (1981). *The Mathematical Experience*. Boston: Birkhäuser.

de Millo, R., Lipton, R., & Perlis, A. (1998). Social Processes and Proofs of Theorems and Programs. In T. Tymozcko (Ed.), *New Directions in the Philosophy of Mathematics: An Anthology* (pp. 201-213). Boston: Birkhäuser.

Devlin, K. (1995). Editorial: Proof Beyond Reasonable Doubt. *Focus–The Newsletter of the Mathematical Association of America*, 15(1), 2-3.

Dreyfus, T. (1990). Advanced Mathematical Thinking, *Mathematics and Cognition: a Research Synthesis by the International Group for the Psychology of Mathematics Education* (pp. 113-134): Cambridge University Press.

Elby, A. (1999). Another reason that physics students learn by rote. *American Journal of Physics, Physics Education Research Supplement*, 67(7 Supp 1), S52-S57.

Fischbein, E., & Kedem, I. (1982). Proof and certitude in the development of mathematical thinking. Paper presented at the *Sixth International Conference of the Psychology of Mathematics Education*, Antwerp, Belgium: Universitaire Instelling Antwerpen.

Franklin, J., & Daoud, A. (1988). *Introduction to Proof in Mathematics*. Englewood Cliffs, New Jersey: Prentice Hall.

Galbraith, P. (1981). Aspects of proving: a clinicial investigation of process. *Educational Studies in Mathematics*, 12, 1-29.

Goetting, M. M. (1995). *The college student's understanding of mathematical proof.* (Vol. 56-A, 3016). University of Maryland: Dissertations Abstracts International.

Grabiner, J. (1998). Is Mathematical Truth Time-Dependent? In T. Tymozcko (Ed.), *New Directions in the Philosophy of Mathematics: An Anthology* (pp. 201-213). Boston: Birkhäuser.

Hanna, G. (1983). *Rigorous Proof in Mathematics Education*. Toronto: OISE Press.

Hanna, G. (1989). Proofs that prove and proofs that explain. Paper presented at the *International Group for the Psychology of Mathematics Education*, Paris.

Hanna, G. (1990). Some Pedagogical Aspects of Proof. In G. Hanna & I. Winchester (Eds.) *Creativity, Thought and Mathematical Proof* (Special issue of *Interchange: A Quarterly Review of Education*). (pp. 6-13) Toronto: Ontario Institute for Studies in Education.

Harel, G., & Sowder, L. (1998). Students' Proof Schemes: Results from Exploratory Studies. In A. Schoenfeld, J. Kaput, & E. Dubinksy (Eds.), *Research in Collegiate Mathematics Education, III* (Vol. 7, pp. 234-283). Providence, RI: American Mathematical Society.

Hersh, R. (1997). *What is Mathematics, Really?* Oxford: Oxford University Press.

Hoyles, C., & Healy, L. (1999). *Justifying and Proving in School Mathematics* (End of Award Report to ESRC). London: Institute of Education, University of London.

Hughes-Hallett, D., & al., e. (1998). *Calculus*. (Second ed.). New York: John Wiley.

Kleiner, I. (1991). Rigor and Proof in Mathematics: A Historical Perspective. *Mathematics Magazine*, 64(5), 291-314.

Kline, M. (1972). *Mathematical Thought: From Ancient to Modern Times*. Oxford: Oxford University Press.

Knuth, E. (in press). Secondary school mathematics teachers' conceptions of proof. *Journal for Research in Mathematics Education*.

Lagasse, P., Goldman, L., & Hobson, A. (Eds.). (2001). The Columbia Encyclopedia. New York: Columbia University Press.

Lakatos, I. (1978). *Mathematics, Science, and Epistemology.* Cambridge: Cambridge University Press.

Martin, G., & Harel, G. (1989). Proof frames of preservice elementary teachers. *Journal for Research in Mathematics Education*, 20(1), 41-51.

Mason, J., Burton, L., & Stacey, K. (1985). *Thinking Mathematically.* (Revised Edition). Reading, Massachusetts: Addison-Wesley.

Moore, R. (1994). Making the Transition to Formal Proof. *Educational Studies in Mathematics*, 27, 249-266.

Movshovitz-Hadar, N. (2001). The Notion of Proof. In L. Grinstein & S. Lipsey (Eds.), Encyclopedia of Mathematics Education . New York: Routledge-Falmer.

NCTM. (2000). *Principles and Standards for School Mathematics*. Reston, VA: National Council of Teachers of Mathematics.

Schoenfeld, A. (1985). *Mathematical Problem Solving.* Orlando: Academic Press.

Schoenfeld, A. (1994). Reflections on Doing and Teaching Mathematics. In A. Schoenfeld (Ed.), Mathematical Thinking and Problem Solving (pp. 53-70). Hillsdale, NJ: Erlbaum.

Sfard, A. (2000). On reform movement and the limits of mathematical discourse. *Mathematical Thinking and Learning*, 2(3), 157-189.

Simon, M., & Blume, G. (1996). Justification in the Mathematics Classroom: A Study of Prospective Elementary Teachers. *Journal of Mathematical Behavior*, 15, 3-31.

Skemp, R. (1971). *The Psychology of Learning Mathematics*. London: Pelican.

Steiner, M. (1978). Mathematical Explanation. *Philosophical Studies*, 34, 135-151.

Stewart, J. (1998). *Calculus: Concepts and Contexts*, 3rd ed. Pacific Grove, CA: Brooks/Cole Publishing Company.

Thurston, W. (1994). On Proof and Progress in Mathematics. *Bulletin of the American Mathematical Society*, 30(2), 161-177.

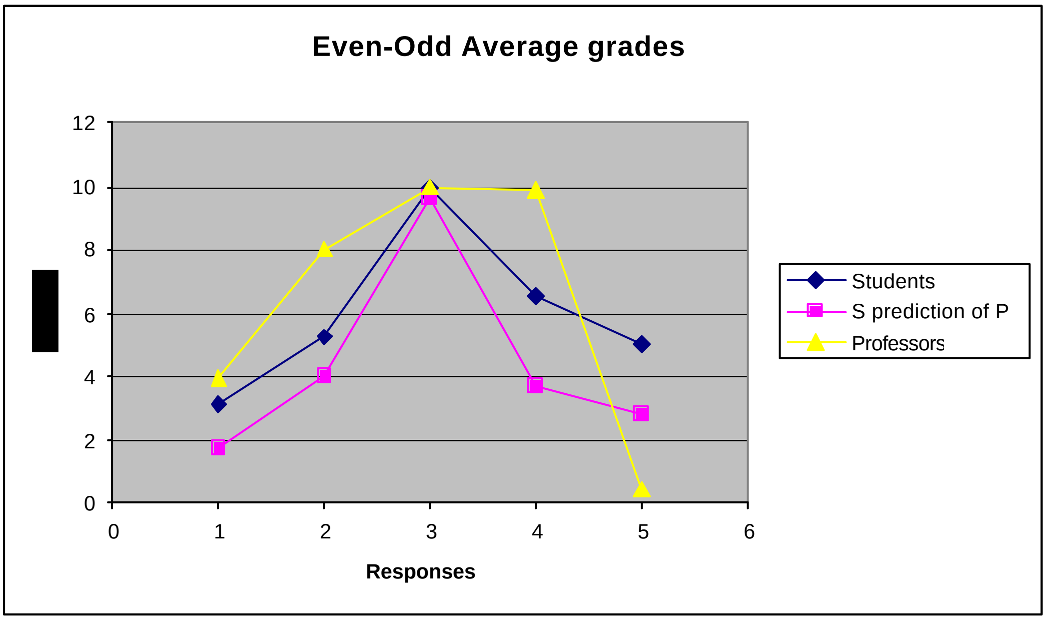
Vinner, S. (1983). The notion of proof: Some aspects of students' view at the senior high level. Paper presented at the *Seventh International Conference of the Psychology of Mathematics Education*, Rehovot, Israel.

Wilder, R. (1944). The Nature of Mathematical Proof. *American Mathematical Monthly*, 51, 309-323.

Appendix A: Results used in second round questioning

1. The Big Picture

Below is the graph reflecting how students and teachers (combining graduate students and professors) graded the different responses to the even-odd problem. I am interested in your reactions.



• What strikes you as interesting? surprising? not-surprising? Why?
• There seem to be big differences between student and teacher perceptions. How would you explain these differences?

2. Individual comments

Below are some student, graduate student, and faculty responses from the first round of interviews.
• Do any responses surprise you? Why or why not?
• Which (if any) of the responses reflect what you perceive as typical views from that group? Why or why not?
• To what extent do the responses from the group that you are in reflect your views? Your student/teachers views?

Note: "I" stands for interviewer. The other letters designate participants. There is a different letter for each excerpt which may or may not be a different person.

Response 1

Student

A: Ok. The first one, is pretty much what I was going for. It is pretty convincing, except it only shows examples. It doesn't really show why. So, yeah, probably on a test it wouldn't get many points because I know this (unintel) is not for examples. He wants proofs, like, oh let x equal this, show why. So I understand that example

I: So if he was giving points out of 10, what would he give?

A: 2 (laugh) for that one, yeah.

I: And what would you give.

A: Like a 5 or 6 probably. Cause I could understand it. I still wouldn't understand why its an even function or odd function. I don't think it really answered the question. It just said give examples, it didn't really answer the question.

B: (reads #1 out loud) I'd give this zero. There's no, like, proof that if you continue on, like for these integers that you will see this pattern. It is just not really solid. These are just examples. There could be a counterexample along the line and you wouldn't have checked for it because you haven't gone through infinite number of integers. And you can't do that. So you haven't proved it for all n.

I: Is it convincing at all to you?

B: No. I just see it as an observation. That, oh, I can take the derivative. This is odd, this is even. It doesn't convince me. Plus (laugh) they don't even state what an odd function really is. They just said, ok, that's an even power that's an odd power so that's even odd. That is not convincing.

Graduate Student

C: And response 1 is (laugh) maybe they'd get a little credit, some partial credit a couple of points for just trying to see what's going on. But it should be emphatically pointed out that there are other functions aside from polynomials.

Faculty

D: Well the first one is suggestive, but not convincing. It is certainly not a proof. It shows the student understands the statement of the question, but doesn't understand what the nature of the proof is. Nonetheless within that range, I think it is fairly close. They don't just take a finite number of examples, they… the student also generalizes and says this is true for every power. And that is getting closer to something that is a proof. You could get it for all polynomials, for example by… pretty readily from this. And indeed if you had Taylor Series you could get it for analytic functions.

Response 2

Student

I: How many points do you think this would get on an exam, out of 10?
E: Out of 10? Here or how many do you think it should get?
I: Both.
E: Here, I think out of 10 it would get about 3. Here I think they really want you to do some equation or some sort of symbol manipulation. And it says to prove. It proves it for all even functions. A statement, you know, that at the end you can state that it proves it for all even functions.

F: Um, ok, for this one, um I don't really think it is very convincing. It doesn't really state anything. It just has a picture. It just basically tells what the picture does. And again it doesn't explain why. It doesn't answer the question. It is less convincing to me because I can't really see an example. It is just, oh, (unintell) the picture over. On the scale from 1 to 10, I would guess it would probably be a 1.
I: Is that what your professor would give or you would give?
F: I would give a 1. The professor would probably give maybe a 3. Cause he likes the, well if and then and which and therefore.

Graduate Student Instructor

G: It (#2) is a fine argument and it can be rigorized very easily. In fact it is quite rigorous. It is in essence nothing more than this (points to his formal proof, like #3). Except it has intuition, so it should get full credit, I think. It has intuition and it's clear what is going to happen when you take. Ok, so maybe you might want to actually write the definition of the derivative down. So, I am not quite sure. Somewhere between 9 and 10 points. I would probably give it 10 if it is in a calc 1A class I would give them full credit.

Professor

H: The second one. Well ok, this is quite a lot better in the sense that there is a real proof in the reach of what is written there. A mathematician at least could look at this and could write out a correct proof, because it is clear that there is an idea there.

Response 3

Student

J: So they proved it (#3) I think. I would give them full credit, 10 out of 10.
I: And what would your professor give?
J: I also think the professor would give 10 out of 10. As long as he understands a little bit more, because I got lost a little bit. But, yeah, I think he'd give a 10 out of 10 also. So in this case, the understanding by a graph isn't necessary. I mean, not completely necessary.

I: What makes this (#3) more convincing than this (#2)?
K: Because it actually shows why. It basically, this one just has a picture, and it doesn't really give any basis for the proof. This one (#3) shows exactly why the definition is correct. I guess you could have added the two. Like, um, put the picture and then you know use the definition of the derivative and then stated that at the very end, combined them. But just the picture alone doesn't say much about the answer.

Graduate Student Instructor

L: And if someone did this proof (#3) they would get full credit too. I might make a remark, like you could draw a picture too. So the true proof would be this and a picture. They would get extra brownie points or something.

Professor

M: but I think I do not like answers like #3 and 4. And if someone is willing to give something like this, I would check it formally. I certainly would not encourage these kinds of responses.

Response 4

Student

N: Is that true? (reads again) (laughs) Wait a minute! Is that true? f of x is even. So f of x equals negative x. Ok, that's true. "Take the derivative of both sides. f'(x) = -f(-x) by the Chain rule. So f(-x) is odd." Um... I would want them to say, like "f prime of x is odd because an odd function states that" what is it "negative f prime of x equals f prime of x." What is it again? f prime of negative x equals negative f prime of x. Well, after this I would like them to state what an odd function is.

I: So how many points do you think it would get?

O: This one, probably like 3, because they state the given and what it is and they give the definitions and they say what they do, but they don't really show any work so we don't know how they arrived at the answer.

Graduate Student Instructor

P: Well certainly response 3 and 4 show at least to me perfect understanding.

Professor

Q: Ah. This is much nicer solution, assuming it is right. It's fine too. That's obviously a slicker solution. The reason I was hesitating a bit was that I was slightly worried about setting this equation and taking the derivative of it. And I would have been happier about that. I would have not worried that there was some error being slipped in they had said let g of x equal f of minus x. And then it's clear it is ok. But it seems to be fine. So, that's great. Better than my solution. And that's also a 10.

Response 5

Student

R: Ok. For response 5, I'm just gonna go… huh? Um… It seems like all they did was just you know put algebra and try to figure out something. But when they said substitute this f of x equals that, it doesn't seem right. Because they put f of negative x equals f of x, but they have the same thing for both sides, so it would just be like f of x equals you know f of negative f of x, and that's like not what they wanted. So, I don't think that's a very good response at all. I mean they show their work, but it doesn't seem right. They got to the right answer, but it seems kind of fishy. I don't know, the professor wouldn't give very many points, probably like 3, 2 or 3 points. I would probably give it like 1 or 2 points.

Graduate Student Instructor

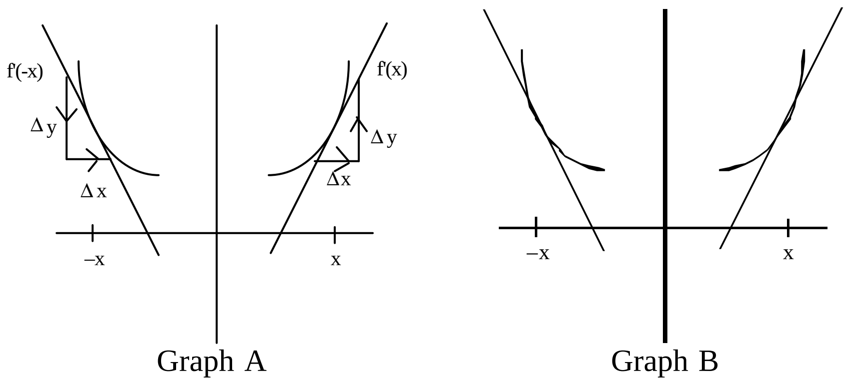
S: What is most important to me is that they should know how to deal with functions. They cannot just factor in this negative one as they say they can. So I would definitely give zero for this one.

Professor

T: Ok. So they have assumed the function is linear here, which means it can't be even unless it is zero. So the proof works. But I think in fact what they have proven is that even functions are identically zero, so their derivatives are identically zero, so those are odd as well. So, what would I do about this? (40 sec) I would give this about a 2. And this would be an example where the student would be very unhappy about the grade. And they would first of all would have some trouble understanding why you couldn't do that. And that could be explained, and they would understand that, but then they would say "that's only… that's an arithmetic mistake so you can't take 8 points out of 10 for an arithmetic mistake." Um… and you would respond, "but there is no correct proof that this is close to." It's not like they were doing a correct idea and they just made a little blunder in it. I don't think. I don't think there is a correct proof that looks at all like this. So I think at the end, they would still have their 2, and they would be unhappy, and I would be frustrated because they would be expressing the feeling that somehow they had been treated unfairly. And I would be frustrated that they didn't understand that what they did was fundamentally wrong rather than wrong in a detail.

3. Variations on responses to even/odd problem

A. Consider the following two graphs:



Would having Graph B instead of Graph A for response #2 change your evaluation of response #2? If so, how would it change? If not, why not?

B. How many points out of 10 would you give the following response? How many would your student/teacher give?

Let f(x) be even, so

$$f(x) = a_n x^{2n} + a_{n-1} x^{2n-2} + ... + a_2 x^4 + a_1 x^2 + a_0$$

Here every power of x is even, so f(x) is even.

$$f'(x) = 2na_n x^{2n-1} + (2n-2)a_{n-1} x^{2n-3} + ... + 2a_1 x$$

Since every power of x is odd then f'(x) is odd.

C. How many points out of 10 would you give the following response? How many would your student/teacher give?

Given f(x) = f(-x). Let g(x) = -x. So f(x) = f(g(x)). Differentiate both sides. f'(x) = f'(g(x))•g'(x) by the chain rule => f'(x) = f'(-x) (-1) =>f'(x) = -f'(-x), so f'(x) is odd.

Appendix B: Paper based on preliminary results

Towards a characterization of proof views held by students and teachers in collegiate calculus

Abstract

The work reported here takes steps towards characterizing the views of mathematical proof held by college freshmen and their two types of teachers—graduate student instructors and mathematics faculty. The data comes from task-based interviews in which people from these groups produce and evaluate responses to a proof-based question from collegiate calculus. A theoretical framework is discussed in which the activities of producing and following a proof are characterized in terms of an interaction between private and public aspects of mathematics. The main empirical result is that while for most teachers the public and private aspects are deeply connected, for students they are not. This disconnect is in part a function of their level of understanding and in part a function of differing epistemological beliefs.

0. Introduction and background

Since the 6^{th} century BC when Greek mathematicians introduced proof as part of the axiomatic method, mathematicians have considered proof to be the hallmark of mathematics. However, the question of what proof really *is* has been, and continues to be, a matter of debate among mathematicians, historians of mathematics, philosophers of mathematics, and mathematics educators. The following caricature of an ideal mathematician, from *The Mathematical Experience* by Davis and Hersh, illustrates part of the debate and some of its implications for teaching.

A student comes and asks our ideal mathematician what is proof. The mathematician responds that a proof is a transformation from a hypothesis to a conclusion using the rules of logic, based on the axioms and the symbols of a formal system. The student is surprised, saying that she has never seen a proof like that in her undergraduate mathematics courses. She pushes the mathematician to explain further. Reluctantly the mathematician gives in and says, "Well, it's an argument that convinces someone who knows the subject." The student points out that that implies proof is subjective. The mathematician stubbornly objects, "No, no! There is nothing subjective about it! Everybody knows what a proof is. Just read some books, take courses from a competent mathematician, and you'll catch on." (Davis and Hersh, pp. 39-41)

However it appears that most students do not simply "catch on." Research indicates that students both at high school and university level have difficulty, not only in producing proofs, but even in recognizing what a proof is (e.g. Chazan, 1993; Moore 1994). The difficulty in understanding the nature of proof has also been reported among prospective elementary school teachers (Simon, 1996) and experienced high school teachers (Knuth, under review). Understanding the nature of proof, in addition to its theoretical interest, seems essential for thinking about how to teach students about proof, both at the university level and throughout the K-12 level, as is dictated by the new NCTM standards (NCTM, 2000).

My dissertation is a study of mathematical proof from both an empirical and a theoretical standpoint. The empirical task is to compare the views of proof held by entering university level students and those held by their two types of teachers—graduate student teaching assistants and mathematics faculty. The claim has been made in passing that university student and teacher views conflict, but there has been little research documenting the nature of the conflict. By looking at students who are just beginning collegiate mathematics, whose views are formed in large part by the views of proof they have developed in high school, the study also sheds some light on differences between high school and university level treatments of proof.

The theoretical task, which is central to completing the empirical one, is to explicate the notion of proof. Following in the tradition of Lakatos (1979), I recognize that proof extends beyond the limited notion of a sequence of logical statements proceeding from axioms to a desired conclusion. Without deciding *a priori* what this broader conception of proof should be, I nonetheless cast my net wide to try to capture both the subjective and objective aspects of proof described by the ideal mathematician above. I end up characterizing proof in terms of a private and a public aspect of mathematics, the interaction between the two being crucial for both being able both to produce a proof and to recognize a proof as a proof.

1. Data collection methods

The data for this study comes primarily from task-based interviews with 11 calculus students, 4 graduate student teaching assistants, and 5 mathematics faculty members from a top-ranked public university. The teachers of the students are included in the participant pool. The task was to prove that the derivative of an even function is odd. Participants were given time to work on the question on their own (which was videotaped) and then asked to explain their work. Next they were shown five different responses to the question (see Appendix). The first response was empirical, the second graphical, the third a long formal proof using the definition of derivative, the fourth a short formal proof using a theorem about derivatives, and the fifth an incorrect formal-looking proof. Participants were asked several questions about the proofs: which is most convincing? why?; how many points (out of 10) would each receive on an exam? why?; which demonstrates the best understanding? why?

In addition to the core interviews, video and/or field note data were collected from lectures and discussion sections which the students attended. Follow-up interviews were also conducted with 2 students, 1 graduate student, and 1 faculty member.

2. Towards a characterization of proof views

2.0 What is convincing?

Part of the problem with characterizing proof as a convincing argument is that "convincing" means different things to different people in different contexts. Saying that a proof is that which is convincing begs the question. This difficulty is illustrated by the following student who demonstrates different senses of conviction first when he worked on a proof and later when he was asked to evaluate one. The first excerpt comes from a discussion about his attempted solution in which he generated examples.

> I: And you said that when you started that you did something to convince yourself that it was true.
> Student A: Right
> I: How convinced are you that it is true? What percent?
> Student A: What percent? I guess, I guess I know it **has** to be true. I mean, I don't know, I mean I guess I know its true, like 100%. I know it has to be true. Because I've done derivatives so many times, you know. But I just couldn't really prove it. Because that's the way it is, but I don't know how to prove such a statement.

The student appears completely convinced at this point in the interview. However a few minutes later he is shown an empirical solution (Response 1 in the Appendix), and in that context he says he is not convinced.

> Student A : The first problem. I'll probably give 3 or 4 points. Granted it has examples, but there are infinite equations to show. This is just not convincing because you're not… you haven't proved anything. You're just showing examples. I mean how to do derivatives... You're not doing anything conceptual. I mean, examples are not proof, so it is not very convincing.

Notice that in the first excerpt, the student distinguishes between convincing and proving ("it has to be true…but I don't know how to prove such a statement"). In the second case he seems to equate the two ("examples are not proof, so it is not very convincing"). This indicates that he uses a different sense of conviction in the first and second cases. In neither case does he equate 'believing something to be true' and 'proving', but the examples that he generates on his own seem to somehow resonate with his experience ("because I've done derivatives so many times you know") in a way that the examples that he is shown in Response 1 do not.

In the following excerpt, Student A is asked to discuss the different senses of conviction himself.

> I: So let me ask you something, before when I asked if you believed it was true from your examples, you said you completely believe it is true, 100%. So how does that relate to what you said now?

> Student A : Well, um, I mean. I guess there is like this doubt in my mind. That it might not be true, but I mean since I know... or I mean if that's how... since I know. There's so many times, I guess… it has to be true… its true… I know it's true. It's true. But, for I mean… we have the same problem because showing examples doesn't mean that its proved, right? You see what I'm saying. You're showing examples.
> I: But I'm just curious about the fact that on one hand you say you **completely** believe its true…
> Student A : Well I don't totally believe its true…
> I: Well, you said 100%, You can revise that if you want.
> Student A : No, I'm not revising it. I still, I'm convinced it is true. But how do you go about proving that's true? It's harder. It's different.

The difference between the two conflicting senses of conviction appears to be robust. Even when given a chance to resolve the conflict, he sticks to both stories. On one hand he is completely convinced by the examples (at least insofar as they resonate with his experience with derivatives of functions). On the other hand he also says he is not totally convinced, and he knows that examples are not proof.

I refer to these two different kinds of conviction as *private* and *public*. The fact that he is 100% convinced comes from a *private* sense of conviction, which is based on:

1. a personal sense of what constitutes mathematical truth in general, and
2. a personal understanding of why a claim is true

The fact that Student A doesn't find response #1 convincing, because he knows examples are not proof, comes from his *public* sense of conviction, which is based on:

3. beliefs about what is considered evidence according to the standards of rigor of a particular mathematical community.

Factor 2 refers specifically to mathematical understanding while factors 1 and 3 refer to beliefs surrounding the doing of mathematics. I will refer to type 1 beliefs as *epistemological* and type 3 beliefs as *social.* Note that epistemological and social beliefs can be different, as they appear to be in this case. The fact that examples do not constitute proof is likely something that this student has heard in his formal mathematics education (hence a social belief). However the belief does not appear to be epistemological since he still holds that the examples really do convince him, 100%, of the truth of the claim. Despite what he has heard in school, his personal sense of what is convincing remains unchanged.

Next we look at senses of conviction held by a faculty member, Prof. A. Prof. A also uses examples to convince himself that the claim is true, followed fairly quickly by thinking about a picture of a generalized even function. (The symbol […] indicates a break in the transcript.)

> Prof. A: First thing is to use examples to convince yourself that it is true. x squared—true, cosine—true. […] Let's see, an even function. There is

> only one thing about it, and that is its graph is reflected across the axis. Yeah, and you can be quite convinced that it is true by looking at the picture. If you said enough words about the picture, you'd have a proof.

He goes on to write out his idea in formulas, which is similar to what is given in Response 4 (see Appendix). He is then asked to compare how convinced he is based on the picture and the formulas. He replies:

> Prof. A : Oh, the first one convinces me completely that it is right. It is right. The second one is how you present it if you want to convince somebody else. It doesn't have… (sigh, look to side) your currency. My currency is kind of… my currency is like pictures. But the general currency that works for everybody is a formula. So if you were looking on a test, you'd be looking for a formula or a lot of words and a (gestures to the picture). But it would take a lot more words to make the picture proof communicatable.

Prof. A, like Student A, seems to have both a private and public sense of conviction ("the first one convinces me completely that it is right. It is right. The second one is how you present it if you want to convince somebody else.") The picture seems to connect with the professor's personal (private) understanding of why the claim is true, but he still feels the picture by itself would be inappropriate for a (public) exam. He does not think the picture by itself necessarily communicates the proof idea.

There seem to be two main differences between the professor and student. One is that the professor is aware of the differences between the public and private realm (the fact that the student vacillates between the two senses of conviction indicates he does not see them as two separate things). The second difference is that the professor is able to map his private ideas (the picture) to a publicly acceptable form (formulas). It is not clear if the student has that ability. He at least says he doesn't know how to ("I don't know how to prove such a statement.")

2.1 What shows why?

Next we will contrast a student's and professor's sense of what shows why the statement is true. Students tended to say the formal proof (#3) shows why while faculty think the picture (#2) does.

> I: What makes this (#3) more convincing than this (#2)?
> Student B: Because it actually shows why. It basically, this one (#2) just has a picture, and it doesn't really give any basis for the proof. This one (#3) shows exactly why the definition is correct. I guess you could have added the two. Like, um, put the picture and then you know use the definition of the derivative and then stated that at the very end, combined them. But just the picture alone doesn't say much about the answer.

While the student talks about combining responses #2 and #3, it seems that the picture for her has only heuristic value. It does not help show why the claim is true. One interpretation of Student B's response is that she does not understand response #2 well enough to recognize that it shows why. In fact, later in the interview, she is asked to explain the picture, and she has difficulty doing so. The excerpt above, then, seems to reflect that the student, not having developed her own private understanding of the proof, relies on the public criteria for an mathematical argument (which will be discussed below, under "what is mathematical").

Compare Student B's comments with those of her professor, Prof. B. He says that while #2 is not a real proof, it is "within reach" of a real proof.

> Prof. B: A mathematician at least could look at this and could write out a correct proof, because it is clear that there is an idea there.

The picture is not just a heuristic to the Prof. B as it was for Student B. Even though he is not willing to call #2 a proof (several others teachers were), he says that the picture captures the 'idea' of the proof. What makes it proof-like is that that this privately held idea can be mapped to a public argument with the appropriate level of rigor.

Like Prof. A, Prof. B distinguishes between private and public aspects of proving, which in this case come in the form of a distinction between an argument that demonstrates 'why' the claim is true and an argument which constitutes a formal proof (an example of what Steiner (1978) refers to as a routine distinction between proofs that merely demonstrate and proofs that explain.)

> I: What kind of response do you think would demonstrate the best understanding?
> Prof. B: Depends on understanding of what. Understanding of why even functions have odd derivatives, in some sense the picture does that best. In terms of understanding what constitutes a proof 3 and 4 are better at that and much less good at understanding why even functions have odd derivatives.

So unlike Student B, Prof. B sees 'showing why' and 'proving' as different activities.

2.2 What is mathematical?

Why do students and faculty make different valuations of what shows why? Part of the reason, as mentioned before, is their level of understanding. Many students have difficulty following proof #2. However, there is also a matter of their epistemological beliefs—their views about what constitutes mathematics. As Schoenfeld (1985) has noted, students and faculty seem to have different perceptions of what counts as thinking mathematically. For the students, thinking mathematically involves algebraic tricks (Student C) and formal language (Student B):

> Student C: Let's see... think mathematically... yeah, that's one thing I thought I didn't have when I took math 1B. I couldn't really think in mathematical terms. I guess thinking mathematically is seeing... like...

> oh...being able to manipulate, if you have some function, you can turn it into another function using all sorts of little tricks. If you are thinking mathematically then you have all these tricks in your head then you could pull it from your pool and utilize it. Yeah.
>
> Student B: (about response #2) I would give a 1. The professor would probably give maybe a 3. Cause he likes the, well 'if' and 'then' and 'which' and 'therefore'.

Consider in contrast the remarks of the following graduate student teaching assistant. He sees the formal aspect of mathematics as "details", which while important, do not really capture what mathematics is about.

> Grad. A: But again, later on, (after calculus) the more math you do the more you realize your intuition can go wrong. So you need to have somehow this checking point. This rigor that needs to check your intuition. But on the other hand you shouldn't let it go so far that all you care about are details, because then you're not doing math.

The fact that students tended to judge a proof by superficial aspects, such as whether it had the right language or amount of formalism, does not distinguish them from their teachers. One famous professor 'fell' for the false-formal proof because he thought it looked right. And several teachers accepted #3 as a proof without checking the details. The factors that seem to distinguish the students from the teachers were that (1) the faculty, when pushed, were able to move past the superficial level, and (2) students tended not to see that their private ideas about why the claim was true could be connected to a publicly accepted proof.

To illustrate (2), consider the way Student A described his difficulty constructing a proof. He made this comment after he was shown the proofs involving the definition of derivative and a theorem known as the chain rule.

> Student A: You are creating something out of nowhere, when you prove. If you don't use the definition of derivative or you don't have the chain rule, I would say it is pretty impossible to go about proving this problem.

In contrast, most of the teachers, like Prof. A above, talked about the formal proofs (#3 and 4) as translations of their idea.

It seems, then, that as people develop mathematically they develop a different sort of appreciation for rigor. While rigor is publicly sanctioned, it is meaningful and mathematical only insofar as it can be connected to private ideas. One of the graduate students I interviewed, who was only 20 years old but in her second year of graduate school, seemed to be just in the middle of that transition. As she talked about how she would grade the different responses, she was torn about how she would grade the picture proof. She explains:

> Grad. B: (About #2) I would like to give it full credit, but somehow I feel I'm just not allowed to. That a picture isn't good enough. It doesn't look like there is any math written down. I know, that is so stupid.

Her view seems to contrast with that of the professor whom she works for, who would give the picture proof full credit because, in his words, a picture is more fundamental than formulas.

2.3 What makes a good proof?

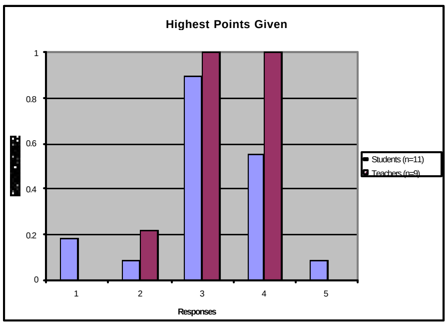
Students have a tendency to value the formal proof (#3) in a way that faculty do not. We saw earlier that some students liked #3 because it explained "why" the claim was true. This was true both of students who did not seem to follow the proof (like Student B above) and those who did (like Student D below).

> Student D: (On #3) Clear as a teardrop. Top of the line. 10 points. It is both convincing and understandable.

In contrast, several teachers did not like proof #3, even though they would give it full credit.

> I: And what did you think about response #3?
>
> Prof. A: (laugh) Wildly too…Way too complicated, but correct in every detail. You get points for correctness even if it is not the shortest proof. […] This is like Stewart. A long proof of something very simple.

We can also see the difference between student and teacher views by comparing the responses for which people gave highest points with the responses which people thought demonstrated the best understanding (see Figure 1).



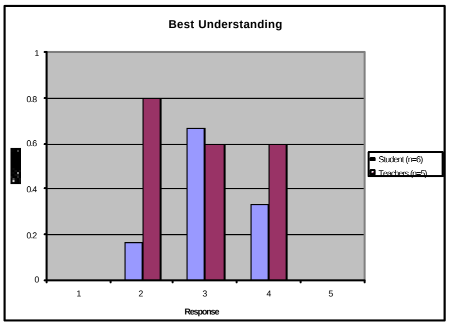
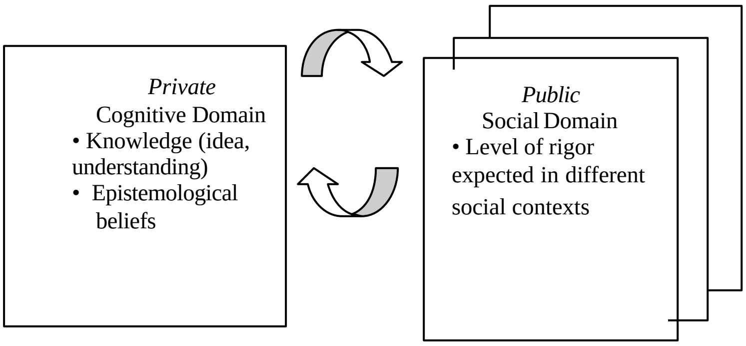


Figure 1

For students (light colored bars) there ion between what they would give highest marks as a good proof and between what they take to show the best understanding (the shape of the bars is the same on both graphs). In contrast, the views of the professors (dark colored bars) is the opposite. What they take to show the best understanding is the opposite of what they would give the highest points.

3. Summary

The following picture captures the key ideas discussed above.



Proof Framework

Proof involves both a public and priv e aspect described here as the cognitive domain, consists of knowledge (a personal understanding of why a claim is true) and epistemological beliefs (a personal sense of what constitutes mathematical truth in general). The public aspect described as the social domain involves recognizing what level of rigor is appropriate for certain social situations. The different boxes on the right hand side of the picture indicate the different social situations (producing an answer on an exam, discussing with a peer, writing for a journal, etc.). Producing or following a proof involves an interaction between these domains, as indicated by the arrows.

This picture helps to capture what appears to be a subtle, but fundamental, difference in how the professors and students in this study view proof. The professors are able to navigate around the entire picture, mapping their ideas into rigorous representations and interpreting representations in terms of mathematical ideas. They see proof as a mapping between the private and public domains—the private containing, in part, the core mathematical idea or ideas and the public containing the representation of those ideas. They see the domains as separate (e.g. my currency vs. everyone's currency) but deeply connected. Both the ideas that are generated in the private domain and those communicated in the public domain count as mathematical. The graduate students are also able to navigate around the entire picture, but one significant difference is that some graduate students (like Grad. B above) are not willing to accept ideas generated in the private domain as mathematical. In contrast with the teachers, the students appear to have a disconnect between the public and private domains. This is in part because of their knowledge—they do not have the idea, and in part because of their epistemological beliefs—they do not expect that ideas generated in the private domain can correspond to anything mathematical.

References

Chazan, D. (1993). High School Geometry Students' Justifications for Their Views of Empirical Evidence and Mathematical Proof. *Educational Studies in Mathematics, 24*, 359-387.

Davis, P, and Hersh, R. (1981) The Mathematical Experience. Boston: Birkhäuser.

Knuth, E. (Under review). Secondary School Mathematics Teachers' Conceptions of Proof.

Lakatos, I. (Ed.). (1976). *Proofs and Refutations*. Cambridge: Cambridge University Press.

NCTM. (2000). *Principles and Standards for School Mathematics*. Reston, VA: National Council of Teachers of Mathematics.

Moore, R. (1994). Making the Transition to Formal Proof. *Educational Studies in Mathematics, 27*, 249-266.

Simon, M. (1996). Justification in the Mathematics Classroom: A Study of Prospective Elementary Teachers. *Journal of Mathematical Behavior, 15*, 3-31.

Steiner, M. (1978). Mathematical Explanation. *Philosophical Studies*, *34*, 135-151.

Schoenfeld, A. (1985). *Mathematical Problem Solving*. Orlando: Academic Press.

Appendix

The Task Prove that the derivative of an even function is odd

Response 1:

Consider the following functions and their derivatives --

$f(x) = x$ odd
$f'(x) = 1$ even

$f(x) = x^2$ even
$f'(x) = 2x$ odd

$f(x) = x^3$ odd
$f'(x) = 3x^2$ even

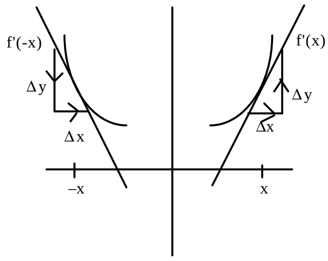
$f(x) = x^4$ even
$f(x) = 4x^3$ odd

$f(x) = x^5$ odd
$f'(x) = 5x^4$ even

$f(x) = x^6$ even
$f'(x) = 6x^5$ odd

Note that for all even functions, the derivative is odd. We could continue for all powers (n = 7, 8, 9, ...), thus the claim is proved.

Response 2:



If f(x) is an even function it is symmetric over the y axis. So the slope at any point x is the opposite of the slope at -x. In other words f'(-x) = - f'(x), which means the derivative of the function is odd.

Response 3:

Want to show if f(x) = f(-x) then - f'(x) = f'(-x).

$f'(-x) = \lim_{h \to 0} \frac{f(-x+h) - f(-x)}{h}$ by the definition of derivative

$= \lim_{h \to 0} \frac{f(x-h) - f(x)}{h}$ since f is even

Let t=–h

$= \lim_{t \to 0} \frac{f(y+t) - f(y)}{-t}$

$= -\lim_{t \to 0} \frac{f(y+t) - f(y)}{t}$

= –f'(x) as desired.

Response 4:

Given f(x) is even, so f(x) = f(-x). Take the derivative of both sides. f'(x) = -f'(-x) by the chain rule. So f'(x) is odd.

Response 5:

f is even, so f(x) = f (-x).
Multiply both sides by -1
- f(x) = - f(-x)
Factor in -1
f(-x) = - f(-x)

Response 5, cont.

f is even so we can substitute f(-x) = f(x)
f(-x) = - f(x)
Take the derivative of both sides
f'(-x) = - f'(x)

So f' is odd, as required.