THE MENTAL MODELS THEORY OF DEDUCTIVE REASONING: IMPLICATIONS FOR PROOF INSTRUCTION[1]

Andreas J. Stylianides — University of Oxford, U.K.
Gabriel J. Stylianides — University of Pittsburgh, U.S.A.

There are currently increased efforts to make proof central to school mathematics throughout the grades. Yet, realizing this goal is not easy, as it requires that students master several abilities. In this article, we focus on one such ability, namely, the ability for deductive reasoning. We first offer a conceptualization of proof, which we use to delineate our focus on deductive reasoning. We then review Johnson-Laird's mental models theory – a well-respected psychological theory of deductive reasoning – in order to enhance what is currently known in mathematics education research about deductive reasoning in the context of proof.

INTRODUCTION

There are currently increased efforts to make *proof* central to school mathematics throughout the grades (e.g., Ball & Bass, 2003; NCTM, 2000). Yet, realizing this goal is not easy, as successful engagement with proof requires that students master several abilities, such as the ability to recognize the need for a proof (e.g., Boero et al., 1996; Mason et al., 1982), the ability to understand the role of definitions in the development of a proof (e.g., Mariotti & Fischbein, 1997; Zaslavsky & Shir, 2005), and the ability to use deductive reasoning (e.g., Foltz et al., 1995; Polya, 1954). In this article, we focus on the ability for *deductive reasoning.*

Available mathematics education research on proof offers: (1) existing evidence that, and insights into how, supportive classroom environments can enable even elementary school students to use deductive reasoning to construct arguments and proofs (e.g., Ball & Bass, 2003; Maher & Martino, 1996); (2) understanding of common difficulties that students face in using deductive reasoning in the context of proof (e.g., Coe & Ruthven, 1994; Hoyles & Küchemann, 2002); and (3) understanding of social and cognitive factors that play a role in students' ability to use deductive reasoning in the context of proof (e.g., Balacheff, 1991; Boero et al., 1996). The findings of this research can be complemented by the findings of psychological research on cognitively guided ways to enhance acquisition of this ability, such as the research associated with Johnson-Laird's (1983; Johnson-Laird & Byrne, 1991) *mental models theory*. Although this psychological research makes only few connections to the notion of proof, it can offer useful insights into proof instruction in school mathematics. Of course, incorporating findings of psychological studies on deductive reasoning into useful practices for promoting students' ability for proof requires first a considerable amount of interdisciplinary research to build necessary bridges between psychology and mathematics education.

Our objective is to review Johnson-Laird's mental models theory – a well-respected psychological theory of deductive reasoning – in order to enhance what is currently known in mathematics education research about this ability in the context of proof. The article is structured into two sections. In the first section, we offer a conceptualization of the meaning of proof in school mathematics and we use this conceptualization to delineate our focus on deductive reasoning. In the second section, we review the mental models theory of deductive reasoning and we discuss implications of this theory for proof instruction.

PROOF AND DEDUCTIVE REASONING

Our conceptualization of the notion of proof in this article is summarized as follows:

> *Proof* is a *mathematical argument*, a connected sequence of assertions for or against a mathematical claim, with the following characteristics:
>
> 1. It uses statements accepted by the classroom community (*set of accepted statements*) that are true and available without further justification;
> 2. It employs forms of reasoning (*modes of argumentation*) that are valid and known to, or within the conceptual reach of, the classroom community; and
> 3. It is communicated with forms of expression (*modes of argument representation*) that are appropriate and known to, or within the conceptual reach of, the classroom community. (Stylianides, 2007, p. 291)

The conceptualization of proof breaks down each mathematical argument into three major *components*: the set of accepted statements (e.g., definitions, axioms, theorems), the modes of argumentation (e.g., application of logical rules of inference like modus ponens), and the modes of argument representation (e.g., verbal, pictorial, algebraic). The use of the terms "true," "valid," and "appropriate" in the conceptualization should be understood in the context of what is typically agreed upon in the field of mathematics nowadays. Of course, this is not to say that these terms have universal meaning in the field of mathematics nowadays, but it is beyond the scope of this article to elaborate on this issue.

The notion of deductive reasoning corresponds to the component *modes of argumentation*. The conceptualization denotes that such modes used in an argument that qualifies as a proof need to be valid and, therefore, they need to support logically necessary inferences from a given set of premises. According to commonly accepted notions of deductive reasoning, logically necessary inferences directly implicate the use of deductive reasoning. For example, Klaczynski and Narasimham (1998) note that deductive reasoning refers to logically necessary inferences drawn "from a general set of givens or premises" (p. 865). Important to note is that we do not associate the notion of deductive reasoning with particular modes of representation, such as modes that may be characterized as formal versus informal.

MENTAL MODELS THEORY AND INSTRUCTIONAL IMPLICATIONS

Before we describe the mental models theory, three caveats are in order. First, although the mental models theory is well respected in cognitive psychology, by choosing it for presentation in this article we do not suggest that it is the best theory currently available in the domain of deductive reasoning. For example, there is Rips's (1994) *theory of deductive reasoning*, which is based on natural-deduction systems for predicate logic and has levels of empirical support similar to the mental models theory. Second, the mental models theory provides an explanation of reasoners' thinking processes on a small range of deductive reasoning tasks, namely, syllogistic inference tasks. Other theories address different kinds of deductive reasoning tasks. For example, the *pragmatic reasoning schema theory* (Cheng & Holyoak, 1985) addresses selection tasks. Third, there is still much to be learned about the mental models theory and how it relates to other theories of deductive reasoning. For example, although we have comparisons of the mental models theory and the pragmatic reasoning schema theory (e.g., Moshman, 1998), psychologists have not yet analyzed fully the relationship between these two theories.

To conclude, our discussion of the mental models theory is intended to initiate discussions and interdisciplinary efforts on how proof instruction can benefit from, and use the findings of, psychological theories of deductive reasoning. We see our discussion as the very first step in a long process that will consider other theories besides the one considered in this article.

Presentation of the theory

The mental models theory assumes that deductive reasoning, as it applies to syllogisms (i.e., arguments from premises to an inference or a conclusion), depends on three main stages (Johnson-Laird & Bara, 1984). First, the reasoner constructs a mental model of the information presented in the premises of a syllogism, where by "mental model" is meant a representation in the mind that has a structure analogous to the structure of the situation it represents. Second, the reasoner scans this model for an informative conclusion that is true. Third, the reasoner searches for alternative mental models that may lead to refutation of the conclusion (counterexamples). In this approach, developmental changes in the ability for deductive reasoning reflect: (1) improvement of the linguistic competence to comprehend logical terms (e.g., and, or, not, if, none, some, all) in the premises and, thus, of the ability to construct appropriate models of those premises; and (2) advancement in the management of these models due to increase in processing capacity (Johnson-Laird, 1990).

Johnson-Laird and Byrne (1991) argue that "people make deductions by building models and searching for counterexamples" (p. 203). They consider that the ability for deductive reasoning is equivalent to the "capacity to build models of the world, either directly by perception or indirectly by understanding language, and [the] capacity to search for alternative models" (p. 204). According to the mental models theory, the unfolding of these capabilities occurs under the control of innate constraints:

> What develops in childhood is the ability to understand language, the processing capacity of *working memory* (Hitch and Halliday, 1983; Case, 1985), and the meta-ability to reflect on one's own performance. Seven year-olds cannot cope with syllogisms because

> they do not understand quantifiers correctly (see Inhelder and Piaget, 1964). Nine year-olds can cope with one-model syllogisms, but not with more than one model (Johnson-Laird, Oakhill, and Bull, 1986; Acredolo and Horobin, 1987). Their *working memory* appears to lack sufficient capacity to retain alternative models of the premises. (Johnson-Laird & Byrne, 1991, p. 204; emphasis added)

As the excerpt above suggests, the degree of success with which mental model construction and examination can be achieved depends on a person's working memory capacity. In other words, the number of models that are constructed and the figural arrangement of terms that can be made within the premises[2], which constitute the two major factors that determine the difficulty of making inferences, seem to be intimately related to working memory:

> The effects of both number of models and figure arise from an inevitable bottleneck in the inferential machinery: the processing capacity of working memory, which must hold one representation in a store, while at the same time the relevant information from the current premise is substituted in it. (Johnson-Laird, 1983, p. 115)

In general, working memory capacity plays a central role in the theory's successful accounting for patterns of performance in deductive reasoning (Johnson-Laird, 1983). More specifically, errors occur because limitations in working memory capacity make people fail to consider all possible models of the premises that would provide them with counterexamples to the conclusions they derive from their initial models (Johnson-Laird, 1983; Johnson-Laird & Byrne, 1991). In turn, this limits individuals' ability for validation, for it constrains their ability to consider more than one model at a time (Johnson-Laird & Bara, 1984).

Johnson-Laird and colleagues' (1986) experiments with two groups of children (9- to 10-year-olds and 11- to 12-year-olds) provide support to the claim that the ability to solve syllogistic problems is associated with the number of mental models that have to be constructed for the solution of a given problem. In one experiment – where the two groups of children drew conclusions from 20 pairs of syllogistic premises – no child in either group made a correct response to the three-model problems, whereas all subjects (the only exception being one 9-year-old) made at least one correct response to a one-model problem. In another experiment – where 16 11-year-olds were tested on all 64 possible forms of syllogistic premises[3] – the children made only 2% correct responses to the three-model problems, as compared to 26% and 63% correct responses to the two- and one-model problems, respectively. Overall, in both experiments, "performance was best with one-model problems and better than chance with two-model problems" (Johnson-Laird et al., 1986, p. 52); correct responses with three-model problems were virtually non-existent.

Investigations with adults revealed a similar pattern of performance, that is, best performance on one-model problems and worst on three-model problems, with the main difference being that the number of problems of each type that the adults solved correctly was typically larger than the corresponding number for children.

Interestingly, when adults had only 10 seconds to respond to syllogistic premises, their performance dropped to a level almost identical to that of the 11-year-olds (Johnson-Laird & Bara, 1984). Anderson et al. (1996) make an attempt to explain these results by using Case's (1984) ideas about short-term *operating* and *storage* spaces: "while the overall capacity of short-term memory does not increase as a function of development, the effect of practice at tasks results in more efficient use of short-term *operating* space, leaving greater capacity in short-term *storage* space" (Anderson et al., 1996; p. 270). Therefore, "[a]s learners get older, they become more adept at building, maintaining in memory, and testing a transitory mental model" (Anderson et al., 1996; p. 270). Increases in the information storage capacity would clearly be beneficial for learners' capacity to achieve these processes.

Discussion of the theory

The mental models theory has not been applied in the teaching and learning of proof, so the examples one can find in the literature illustrating the theory are not focusing on the notion of proof (rather, they are mostly syllogistic tasks placed in non-mathematical contexts). We begin our discussion of the theory with an example of a proving task that we constructed in order to illustrate possible applications of the theory in the particular domain of proof.

> Consider the following two premises, which are basically definitions for multiples of 3 and 6:
>
> An integer is a multiple of 3 if and only if it is three times an integer.
>
> An integer is a multiple of 6 if and only if it is six times an integer.
>
> What can be said (if anything) about any multiple of 6 in relation to a multiple of 3? Prove your answer.

Using the first premise, the following algebraic expression is constructed for any multiple of 3: $3l$, where l is an integer. Likewise, using the second premise, the following expression is constructed for any multiple of 6: $6k$, where k is an integer. Using the information in the two premises, the following model is constructed:

Any multiple of 6 is of the form $6k = 3 \bullet (2k) = 3l$, a multiple of 3.

As this model cannot be falsified with an alternative model, the conclusion is considered valid. That is, we have proved that any multiple of 6 is also a multiple of 3. Successful completion of this proving task depends, given the tenets of the mental models theory, on the linguistic competence of the reasoner to comprehend the logical term "if and only if" in the premises and on the reasoner's processing ability to combine the information from the two premises and search for alternative models.

The structure of the proving task we just analyzed was purposefully organized so that there is a clear set of premises from which the reasoner can draw information to construct a mental model. Yet, most of the proving tasks encountered by students and professional mathematicians do not have this clear structure. Typical proving tasks consist only of the question/prompt (in the particular case: "What can be said [if anything] about any multiple of 6 in relation to a multiple of 3? Prove your answer."), leaving it up to the reasoner to select a collection of relevant premises from his or her community's set of accepted statements (cf. our conceptualization of proof) to construct a proof. Accordingly, the mental models theory seems to be useful in accounting for the solution of a proving task once a set of premises has been specified (either by the task itself or by the reasoner). Of course, the reasoner can revise the set of premises by adding or deleting premises in order to end up with a sufficient set of premises for the solution of the proving task. Each time a new set of premises is established, the mental models theory can be reapplied.

The mental models theory denotes that limited working memory capacity constrains students' performance in deductive reasoning tasks of which proving tasks are a proper subset. This implies that mathematics educators can potentially foster the improvement of students' performance in proving tasks in two interrelated ways: (1) by preventing unnecessary usage of students' working memory when they engage with proving tasks, and (2) by helping students develop strategies for effective managing of their working memory.

An example of (1) is for mathematics educators to engage students in "scaffolded" proving tasks (like the one presented earlier) that specify for the students a small set of relevant premises for the solution of the task. By excluding irrelevant premises from students' consideration when engaging with a proving task, students are freed from the memory-consuming effort to combine information from more premises and construct more complicated mental models than they actually need to. In this way, students are facilitated to focus on the logical structure of the proof and the ideas involved in it. Of course, at some point, educators would like students to become able to identify by themselves the relevant premises for the solution of a proving task. Yet, the kinds of scaffolded proving tasks described earlier can be very useful in the early stages of students' engagement with proof, for they can help students develop necessary skills that will support their independent engagement with proof in the future.

An example of (2) is for mathematics educators to help students develop the strategy of *representing the information in the premises in equivalent and easier to manage forms* (from a working memory standpoint). Looking back to our analysis of the proving task at the beginning of our discussion of the theory, we see that the two premises were reformulated to algebraic expressions, which, due to their conciseness, reduce the processing load thereby facilitating the solution of the task. A related strategy that mathematics educators can assist students to develop is *making efficient use of visual records* such as diagrams (see, e.g., Bauer & Johnson-Laird, 1993; English, 1998; Sweller et al., 1998). This strategy can involve not only the use of a visual record when one is not already offered in a proving task but also the use of an existing diagram to practically integrate disparate sources of information and facilitate solution. It is often the case in proving tasks, especially in geometry, that students are offered a diagram (e.g., a geometric figure) and then a set of givens (premises) which, although refer to the diagram, are separated from the diagram. To make sense of the two sources of information, the diagram and the givens, students must mentally integrate them. For example, to derive any meaning from a given, students must read the given, hold it in their working memory, and then search the diagram for the appropriate referents. This mental integration process is clearly cognitively demanding and occupies a large part of their working memory capacity.

The works of Case (1984) and Anderson et al. (1996) we reviewed earlier suggest that *practice* can help students become more skillful in overcoming the limitations of their working memory capacity, thereby pointing to an educational implication for improving students' ability for deductive reasoning. This implication has first been proposed by Johnson-Laird (1983) based on his observations of "spontaneous improvement in [deductive] reasoning ability simply as a consequence of practice (with no feedback)" (p. 124). Adults in Johnson-Laird's experiments who have been tested twice within a week showed a 10% improvement in their performance, without even forewarning that they would be retested (Johnson-Laird & Steedman, 1978).

The idea that practice can play an important role in the development of students' ability for deductive reasoning, and thus in their ability for proof, is not emphasized in mathematics education research on proof. Research studies on teaching practices that have successfully promoted students' ability for proof do not explicitly identify practice as one of the factors that might have contributed to this success. Part of the reason for which there has been little attention to the potential role of practice in proof learning might be that practice has often been associated with secondary aspects of students' engagement with proof, such as the writing of a proof in the two-column form. According to the two-column form, which prevailed in high school geometry courses in the Unites States for almost a century, "the statements of the proof [are placed] in steps in a column occupying the left half of the page, and … the reasons of the statements [are placed] in steps at the right side of the page, with each reason directly opposite its statement" (Shibli, 1932, p. 145). The emphasis on the form made the writing of a proof a ritual procedure that had to be practiced and memorized; as a result, "the substance of proof as a logical and coherent chain of reasoning that guarantees that something must be true became obscured" (Schoenfeld, 1991, p. 325).

Yet, associating practice only with secondary aspects of students' engagement with proof does not do justice to the potential role that practice can play in proof learning as suggested by the psychological research reviewed in this article. An obvious possible use of practice is to help students develop the kinds of strategies for effective managing of their working memory that we described earlier. In addition, we hypothesize that practice can be used to help students internalize the general logical structure of different proof methods, such as proof by contradiction, thus releasing working memory capacity to be spent in the application of these proof methods. For example, consider the proposition: "There is no smallest positive rational number." In a proof of this proposition by contradiction, one would start by assuming the opposite of the proposition and would show that it leads to a logical contradiction:

> We assume the opposite of the statement we wish to prove: "There is a smallest positive rational number, say y." Now let $x=y/2$. Then x is a positive rational number that is smaller than y. But this contradicts our initial assumption that y is the *smallest* positive rational number. So we can conclude that the original proposition must be true – "There is no smallest positive rational number."

If a student who has not internalized the logical structure of the proof method by contradiction attempts to apply the method to prove the proposition, this student will experience increased processing load of working memory and will likely face increased difficulties with the proof.

CONCLUSION

In this article, we reviewed an influential and well-respected psychological theory of deductive reasoning, namely, Johnson-Laird's theory of *mental models*, in order to enhance what is currently known in mathematics education research about this ability in the context of proof. Our review offered useful insights into potentially effective instructional practices for fostering students' ability for deductive reasoning in the context of proof. Nevertheless, it is a long way before these insights can find their way to the practices of ordinary teachers.

A major challenge, but also a primary urgency, for researchers concerned with issues of proof instruction is to identify effective ways to synthesize relevant research programs in mathematics education and psychology. This article has made a first step towards this direction by bringing to the attention of mathematics education researchers a rich body of psychological research on deductive reasoning and by identifying important issues that require research attention. An interdisciplinary and collaborative approach to the problem of promoting proof in students' learning of mathematics promises major advancements.

NOTES

1. The two authors had an equal contribution in writing this article.

2. For example, syllogisms that involve two premises with three terms (X, Y, and Z) can occur in one of four figures as shown below:

X – Y	Y – X	X – Y	Y – X
Y – Z	Z – Y	Z – Y	Y – Z

3. The 64 possible forms of syllogistic premises are derived as follows: four quantifier combinations for each of two premises that can occur in one of four figures as explained in endnote 2 (i.e., $64=4^3$).

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