MATHEMATICS LEARNING AND THE DEVELOPMENT OF GENERAL DEDUCTIVE REASONING

Michal Ayalon Ruhama Even

Weizmann Institute of Science

This study aims to examine the approaches of people involved in mathematics education and logic to the role played by learning mathematics in the development of general deductive reasoning. The data source includes 21 individual semi-structured interviews. Analysis based on the Grounded Theory method identified three distinct groups of interviewees with relation to views of (a) the meaning of deductive reasoning, (b) the relationships between logical rules inside and outside mathematics, (c) the aspects of deductive reasoning that can be developed through learning mathematics, and (d) the likelihood of mathematics learning to develop deductive reasoning.

INTRODUCTION

The development of deductive reasoning, not only in mathematics, but in general, is stated as a goal of mathematics teaching in many curricula from all over the world (e.g., National Council of Teachers of Mathematics, 2000; Qualifications and Curriculum Authority, 2006). This study aims to examine the approaches of people who are involved in various aspects of mathematics education and logic to the role played by learning mathematics in the development of general[1] deductive reasoning. Following is a brief review of the literature concerning deductive reasoning – in general, in mathematics, and outside mathematics, and the role of learning mathematics in the development of deductive reasoning.

Deductive reasoning

There are various sorts of thinking and reasoning. Among them are association, creation, induction, plausible inference, and deduction (Johnson-Laird & Byrne, 1991). Deductive reasoning is unique in that it is the process of inferring conclusions from known information (called premises) based on formal logic rules, where conclusions are necessarily derived from the given information and there is no need to validate them by experiments[2]. Valid deductive arguments preserve truth in the sense that if the premises are true, then the conclusion must also be true. An example for a common form of deductive inference is the syllogism, which consists of two premises and an inferred conclusion. For instance, All A are B; Some C are A; Therefore, some C are B. No matter what terms we substitute for A, B, and C, the result is a valid deduction. Thus, the following argument is valid: All kinds of music are enjoyable; punk is a kind of music; therefore, punk is enjoyable. Obviously, not all will agree with this conclusion, but the form of the argument assures us that in the case that the premises are true, the conclusion is true as well.

Deductive reasoning and mathematics

Deductive reasoning is most significant in mathematics. And indeed, deductive reasoning is often used as a synonym for mathematical thinking, especially by the formalist school. The formal mathematical-deductive method is defined as starting with undefined terms, and some unproven statements – axioms or postulates. Other mathematical statements (i.e., theorems) are deduced from them using the rules of formal logic, forming a chain of deductions. In the pure formalist approach statements are neither true nor false because they are about undefined terms. Being free from the need to attend to the truth of mathematical statements enables mathematical explorations not available otherwise. Still, mathematics does not remain in the pure formal level. The undefined terms and axioms are often interpreted in connection to the world in which we live, and truth is associated with these interpretations. In this regard, the axioms of a specific mathematical theory are often said to be true and the theorems deduced from them are then also said to be true (Davis & Hersh, 1981). Deductive reasoning is central to mathematics for proving the truth of mathematical ideas, and for recording these ideas. However, it is commonly accepted in recent years that conjecturing, exploration, and creation of new mathematical objects and ideas are seldom done by deductive reasoning. Rather they are based on inductive and intuitive methods (Eves, 1972; Lakatos, 1976; Polya, 1954), similar to the way science is developed.

Deductive reasoning outside mathematics

Since the early days of Greek philosophical and scientific work, deductive reasoning has been considered as a high (and even the highest) form of human reasoning (Glantz, 1989; Luria, 1976). Still, deductive reasoning plays a different role in science than in mathematics. In contrast with modern mathematics, science strives to describe the real world. The scientific process is based to a large extent on inductive reasoning – developing hypotheses based on empirical observations to describe “truths” or “facts” about our world (Freudenthal, 1977; Popper, 1968). Whereas this process has similar characteristics to the way mathematical conjectures are often developed, the stage of providing evidence for the truth of the conjecture is different. Scientific hypotheses, unlike mathematical conjectures, can only be supported – not proven deductively. Nonetheless, deduction is an important tool in science for refuting hypotheses and also plays a major role in predicting and explaining scientific phenomena (Freudenthal, 1977).

Thus, plausible reasoning, and not deductive reasoning, characterizes science as well as other domains, like law and economics (Polya, 1954). Many suggest that everyday activities are even more remote from deductive reasoning (Duval, 2002, Krummheuer, 1995; Toulmin, 1969). In daily life people do not support their claims by a deductive sequence of derivations. Convincing others in the truth of one's claims (or in the rational of one's choices) is the main concern, and not their validity. Thus, substantial arguments (Toulmin, 1969), which do not have the logical rigidity of formal deductions, but are rather more of the plausible type, are often more used, gradually support a statement or a decision, motivated by the need or desire to convince (Perelman & Olbrechts-Tyteca, 1969).

Developing deductive reasoning via learning mathematics

The essential role that deductive reasoning plays in mathematics, on one hand, and the questionable use of deductive reasoning in other fields, on the other hand, raises several issues related to mathematics education. One of them (to which this study relates) is the question of developing deductive reasoning via mathematics learning. Indeed, curriculum guidelines, textbooks and teacher guides in many countries state that mathematics teaching helps students develop their ability to reason logically, and that one of its goals is the development of deductive reasoning, not only in mathematics, but in general. For example, the *Qualifications and Curriculum Authority* (2006) states: "Mathematics equips pupils with a uniquely powerful set of tools to understand and change the world. These tools include *logical reasoning*, problem-solving skills, and the ability to think in abstract ways" (emphasis added). Similar claims are suggested by several researchers (e.g., Clements & Battista, 1992; Morris & Sloutsky, 1998). For example, Polya (1954, p. v) wrote: "Everyone knows that mathematics offers an excellent opportunity to learn demonstrative reasoning". However, Polya himself challenges the role of demonstrative reasoning in real life situations: "Anything new that we learn about the world involves plausible reasoning, which is the only kind of reasoning for which we care in everyday affairs". Later he continues: "The general or amateur student [one who does not intend to make mathematics his life's work] should also get a taste of demonstrative reasoning: he may have little opportunity to use it directly, but he should acquire a standard with which he can compare alleged evidence of all sort aimed at him in modern life" (p. vi). A question is, then, raised – to what extent should the development of deductive reasoning be part of mathematics education? This study asked for the opinions of people who are involved in mathematics education and logic about the connections between mathematics learning and the development of general deductive reasoning.

METHODOLOGY

The research population includes 21 participants. Most of them, 17, belong to at least one of the following groups: junior-high school mathematics teachers, mathematics teacher educators, mathematics curriculum developers, researchers in mathematics education, and research mathematicians. Two other participants are researchers in science education who study logical thinking, and the remaining two are logicians.

Individual semi-structured interviews were conducted with each one. The interviews lasted one to two hours, and focused on different issues related to the role of learning mathematics in the development of deductive reasoning. The interviews were transcribed. Using the Grounded Theory method (Glaser & Strauss, 1967) we coded the data from the interviews and generated initial categories, which were constantly compared with new data from the interviews. Based on refinement of the initial categories, we identified core categories, and used them as a source for theoretical constructs. Some of the main aspects that were developed through this process are discussed in this paper: The meaning of deductive reasoning, in general, in mathematics and outside it; the aspects of deductive reasoning which can be developed through learning mathematics; and the likelihood of mathematics learning to develop deductive reasoning.

DEVELOPING GENERAL DEDUCTIVE REASONING VIA LEARNING MATHEMATICS

All 21 interviewees who participated in this study argued that learning mathematics could develop general deductive reasoning. They also pointed out that developing deductive reasoning should be one of the objectives of mathematics education. One interviewee, for instance, when asked whether he thinks that improving deductive reasoning is a goal of mathematics education, replied:

> Eventually the instruction of mathematics has two main objectives. One of them is to train those people who will use mathematics, and the other is intended for those who won't use mathematics afterward – to present an example of deductive reasoning… I think that developing deductive reasoning is a very important aim… It is the role of mathematics teaching (interviewee no. 5, a curriculum developer and a teacher educator).

But what do the interviewees mean when saying that learning mathematics could improve deductive reasoning? How likely it is that learning mathematics will contribute to the development of such reasoning? What do they mean when claiming that improving students' deductive reasoning is one of the goals of mathematics instruction? And what is their approach to deductive reasoning, in mathematics and outside it? Analysis of the interviews reveals that the interviewees provide different answers to these questions and attribute different meanings to the following aspects: the meaning of deductive reasoning, its nature in mathematics and outside it, the aspects of it which can be developed through learning mathematics, and the likelihood of mathematics learning to develop deductive reasoning. Three distinct groups of interviewees were identified, with the members of each group consistent in their approaches to each aspect. Below is a short review of the groups' views, accompanied with a few (because of limits of space) illustrative excerpts from the interviewees.

Group A

Four interviewees belong to group A. They describe deductive reasoning as a process in which one develops a solution to a given problem in a systematic, step-by-step manner. Each step of this process is derived from the previous one, and leads to the next. However, no indication was given as to how a step is derived from its predecessor. These interviewees consider the logical rules inside mathematics as identical to those of outside-mathematics thinking. They view logical rules, both inside and outside mathematics, as systematic principles of thinking, according to which thinking progresses step by step.

These interviewees claim that learning mathematics contributes to the improvement of deductive reasoning in the development of systematic habits of mind. They ascribe this development to the systematic structure of mathematics and to the methodical, step-by-step way of solving mathematical problems. According to them, the development of deductive reasoning occurs spontaneously as a consequence of doing mathematics. Doing mathematics provides experiences in, and thus improves students' deductive reasoning. For example, an interviewee was asked whether learning mathematics could improve deductive reasoning. She replied:

> I think that mathematics improves deductive reasoning, and I think that it is one of mathematics' main goals… I know that generally, as I told you, it will teach him [the student] to think logically and will give him tools to think and a desire to think and to be organized and systematic… Just from learning mathematics, his logical thinking develops in other fields in life as well. But I don't want while teaching, in every new theorem or in every new formula I teach him, to ask myself what kind of systematic tool it provides him with… I don't take each thing and filter it through a 'thinking strainer'… It happens by itself (interviewee no. 11, a senior high school teacher).

Group B

Thirteen interviewees belong to group B. They relate deductive reasoning to an action of inference or validation using rules of formal logic. Whereas group A focuses on deductive reasoning as a systematic, step-by-step process, group B members center on characteristics of the transition from one step in the deductive process to the next: they focus on the logic essence of an inference, on its validness according to logical rules. In addition, while group A focuses on deductive reasoning as means of solving a given problem, group B members refer to it as means of building and validating arguments. They assert that logical rules used in mathematics (i.e. formal rules of inference) are also used outside mathematics, for example, when trying to understand the insurance rights one would have according to different levels of price. However, these interviewees claim that different factors affect deductive reasoning outside mathematics. Thus people apply other, usually 'softer' rules of inference, in addition to the rigorous ones. Two distinct opinions regarding the factors that affect reasoning outside mathematics are found among the interviewees: Six of them (group B1) talk about external conditions, such as uncertainty and complexity of phenomena in nature and society. The other seven (group B2) explain the distractive influence by internal conditions, such as emotions and beliefs.

Group B members claim that learning mathematics could develop habits of argumentation (not necessarily deductive). Mathematics, they claim, because of its particular nature of validation, enables the exposure to deductive justifications and validations. Moreover, its relatively abstract, detached from reality nature can provide students with opportunities to learn and to apply logical validation, without the distractive influence of prejudices and beliefs that exists in life. Thus, for example, emphasizing the meaning of proof or the different functions of statements (e.g., given information, claims to be supported), can contribute to the improvement of students' skills of argumentation, such that are also relevant to outside mathematical contexts. The examples given by those interviewees include providing grounded justifications (even if not deductive) for beliefs and knowledge in daily life, or critically examining of the rationality of claims.

Unlike group A, group B members argue that there is a need for a deliberate intervention in the mathematics instruction in order that mathematics contributes to the development of argumentative skills. Some claim that in order to teach mathematics in a way that will improve these skills, logic should be introduced as a separate unit of study within mathematics. Others suggest various ways of emphasizing deduction constantly and continuously, claiming that in order to develop argumentative skills, there is a need to explicitly teach and practice principles of deduction as an integral part of mathematics lessons, in various situations and problems,

> I think that students who are involved with deductions in mathematics, and whose teacher points at deductive connections and at logical mistakes, can improve their deducing ability, and to like, for example, to look for deductive connections or to identify logical fallacies. I know I like doing it. Even if it is not a real deductive argument, but more of the plausible kind, whoever meets deductions in mathematics will be able to make much more rational inferences in his life, not only intuitive ones (interviewee no. 9, a curriculum developer and a teacher educator).

Group C

Four interviewees belong to group C. Like those of group B, they also see deductive reasoning as an action of inference or validation using logical rules. However, they argue that outside mathematical contexts, we do not or even cannot use the formal logic rules existing in mathematics. One reason for that claim was that the essence of thinking inside mathematics is entirely different from that outside it. Another explanation was that in daily life, as opposed to mathematics, one barely encounters suitable circumstances for using logical rules. Some also argued that even if one encounters such an opportunity, it is not likely that s/he applies them, because in everyday discourse specific argumentative norms exist. According to these norms, the logic of an argument that one builds is neither a necessary condition for understanding nor for accepting the argument.

The interviewees of this group believe that learning mathematics may influence students' general deductive reasoning. However, they find it hard to point out in what ways exactly. Moreover, according to them, even if the possibility of promoting students' deductive skills through learning mathematics does exist, it seems difficult to reach, because of the current demands of the educational system, especially the matriculation exams. For example, an interviewee was asked whether learning mathematics can improve deductive reasoning. He replied:

> It is not that I think it impossible to teach deductive reasoning through mathematics. I believe that mathematics has some influence on this thinking. I just don't know what kind of influence, and can't tell how it could be done. And even if we assume that it is possible to do so, I don't believe it is possible in the present system… How can one teach and learn logical thinking if one is facing the pressure of the matriculation exam? (interviewee no. 4, a researcher in mathematics education and a mathematician).

As these interviewees do not offer an alternative system by which mathematics instruction can promote deductive reasoning, they actually leave the question of promoting it via learning mathematics open and with deep reservation. Table 1 summarizes these findings.

Table 1: Summary of findings

	Meaning of deductive reasoning	**Logical rules inside and outside mathematics**	**What learning mathematics develops in deductive reasoning**	**The likelihood that learning mathematics improves deductive reasoning**	**# interviewees n=21**
A	Systematic process	Unification	Habits of mind of systematicness	Spontaneity	4
B	Formal logic based inference	Inclusion (external, internal)	Habits of mind of argumentation	Intervention	13 (6,7)
C		Separation	Cannot point out	Reservation	4

CONCLUSIONS

The findings of this study suggest that all its participants view the development of general deductive reasoning as a goal of mathematics instruction. They all assume that to some degree this goal is attainable. However, differences were found among the participants regarding the likelihood and degree of difficulty of achieving this goal. The differences seem to relate mainly to the participants' approaches to deductive reasoning, in general, in mathematics, and outside it[3]: Some of them describe deductive reasoning as a systematic step-by-step approach for solving problems. Being systematic in thinking is one feature of deductive reasoning, which characterizes other kinds of reasoning as well. It is also something that people come across in diverse non-mathematical situations. Likewise, these interviewees consider the logical rules inside mathematics to be identical to those in outside-mathematics thinking. Consequently, these interviewees may naturally point at the simplicity by which the development of systematic habits of mind occurs through learning mathematics. On the other hand, the interviewees who describe deductive reasoning as an action of inference based on rules of formal logic, attribute, as the literature does, complexity to the nature of deductive inference in different domains of life. Accordingly, they consider the development of deductive reasoning through mathematics learning as a complex process that requires deliberate intervention. Moreover, their referring to the aspects of deductive reasoning which can be developed does not relate exclusively to deduction (they refer to argumentation, but not necessarily to deductive argumentation). Some of these interviewees are even not sure whether the process of developing deductive reasoning through learning mathematics is at all possible.

The fact that most interviewees claim that, to some extent, mathematics learning can, and even should, contribute to the development of deductive reasoning, suggests that this issue deserves further attention. In particular, it would be worthwhile to examine in what sense, and under what conditions, learning mathematics develops (as most interviewees claimed) skills of argumentation. Another issue raised by this study is whether specific sub-communities in the community of mathematics educators tend to approach deductive reasoning and its development through learning mathematics in particular ways. Group A includes two teachers and two curriculum developers, one of which is also a teacher educator. There are several other teachers, teacher educators and curriculum developers in the other groups as well. However, all the mathematicians, the researchers in mathematics education and in science education, and the logicians belong to the other groups (B&C)[4]. Indeed, the relative small size of the research population does not allow generalization. Still, it seems worthwhile to study more thoroughly whether there is a connection between the nature of people's professional activities and their approaches towards deductive reasoning and its development through learning mathematics.

NOTES

1. By 'general deductive reasoning' we mean deductive reasoning that is not restricted to mathematics, but can be implemented in other fields as well.

2. This is the classic approach to deductive reasoning, which is also adopted in this paper. There are also other approaches; the main one is based not on formal rules of inference but on manipulations of mental models representing situations (Johnson-Laird, 1999).

3. For an elaboration of these approaches see Ayalon & Even, 2006.

4. A more refined report on the characterization of the approaches of each type of population is in preparation.

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